Stochastic Dynamics with Singular Lower Order Terms in Finite and Infinite Dimensions

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1. Introduction

Diffusion processes are stochastic processes originally used to describe the physical phenomenon of diffusion. Nowadays they have found vast applications in physics, finance, biology and control problems. Their mathematical theories are firmly based on modern probability theory, or more precisely, Itô calculus. In this work, we mainly use analytical methods to construct diffusion processes with singular drift coefficients.

2. Brownian Motion

The simplest diffusion process is the Brownian motion, which is named after the Scottish botanist Robert Brown. In 1828, R. Brown observed that pollen grains suspended in water move chaotically, incessantly changing their direction of motion. The physical explanation of this phenomenon is that a single grain suffers innumerable collisions with the randomly moving molecules of the surrounding water.

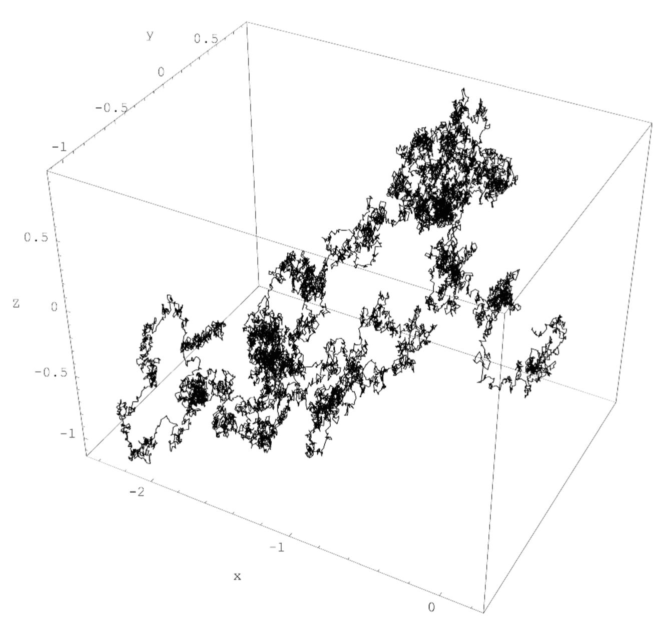

Figure 1: *A single realization of three-dimensional Brownian motion for times* $0 \le t \le 2$.

Brownian motion was put on a firm mathematical foundation for the first time by N. Wiener in 1923. Using the language of modern probability theory, we call a stochastic process W_t a d-dimensional Brownian motion if

- $W_0 = 0$
- W_t has independent increments
- for any $0 \le s < t$, $W_t - W_s$ is normally distributed with mean zero and covariance matrix equal to $(t-s)I_d$, where I_d is the $(d \times d)$ identity matrix.

3. Brownian Motion With Singular Drift: Analytic Approach

In 1944, K. Itô first used stochastic calculus to study diffusion processes. Using Itô's theory, it is possible to characterize the infinitesimal motion of a diffusion particle. The dynamics of a diffusion particle in $\mathbb{R}^d$ is usually governed by a stochastic differential equation. As an example, we could look at a stochastic differential equation of the following simple type:

$$\begin{cases} dX_t = dW_t + b(t, X_t)dt, & t \ge s \\ X_s = x \end{cases} \qquad (1)$$

where $b(t,x) : [0,\infty) \times \mathbb{R}^d \to \mathbb{R}^d$ is measurable and W_t is a d-dimensional Brownian motion. Because of intuitive physical meanings, $b(t,x)$ is usually called the drift coefficient.

The (weak) solution to (1), if it exists, is usually called Brownian motion with drift b starting from (s,x).

Problem: Can we solve (1) with singular drift term $b(t,x)$?

There has been a lot of work on this problem (cf. [1,2,3,4]). Here we want to emphasize the analytical approach. The idea is to we use PDE theory to construct the transition density function (heat kernel) $p(s,x;t,y)$ for the desired process X_t.

More precisely, we could look at Kolmogorov's backward equation:

$$\frac{\partial u}{\partial s} + \frac{1}{2}\triangle u + b(s,x) \cdot \nabla u = 0. \qquad (2)$$

If $b(s,x)$ is smooth and has compact support, it is well-known that for (2) there exists a classical fundamental solution which is exactly the transition density function for the diffusion process described by (1).

When $b(s,x)$ is merely bounded and measurable, classical fundamental solutions for (2) do not exist in general. However, D.G. Aronson's work (cf. [5]) tell us there still exists a fundamental solution $p(s,x;t,y)$ for (2) in a weak sense when $b(s,x)$ only satisfies some integrability condition.

Using this weak fundamental solution $p(s,x;t,y)$ as the transition probability density of the desired process, N.I. Portenko constructed a weak solution to (1) for a broad class of drift coefficients (cf. [2]).

Recently, R. Bass and Z. Chen used another method to solve (2) (cf. [3]). They proved that if the drift $b(t,x)$ is independent of time (i.e. $b(t,x) = b(x)$) and each component $b^i(x)$ belongs to the Kato class $\mathcal{K}_{d-1}$, namely

$$\lim_{r\to 0} \sup_{x\in\mathbb{R}^d} \int_{B(x;r)} \frac{|b^i(y)|}{|x-y|^{d-1}} dy = 0,$$

then (1) has a unique weak solution. Their method is based on constructing the resolvent S^λ of the desired process described by (1).

4. My new results

We study the stochastic differential equation

$$\begin{cases} dX_t = dW_t + b(t, X_t)dt, & t \ge s \\ X_s = x \end{cases} \qquad (1)$$

with a new class of time-dependent singular drift terms. Here we only consider weak solutions to (1). It is well-known that existence and uniqueness of weak solutions to (1) is equivalent to the martingale problem for the operator L being well-posed, where

$$L_t = \frac{1}{2}\triangle + b(t,x) \cdot \nabla$$

We assume $|b(t,x)|$ to be in the forward-Kato class $\mathcal{FK}^{\alpha}_{d-1}$ for some $\alpha < \frac{1}{2}$, namely

$$\lim_{h\to 0} N_h^{\alpha,+}(|b|) = 0,$$

where

$$N_h^{\alpha,+}(|b|) := \sup_{(s,x)\in[0,\infty)\times\mathbb{R}^d} \int_s^{s+h} \int_{\mathbb{R}^d} \frac{1}{(t-s)^{\frac{d+1}{2}}} \exp\left(-\alpha\frac{|x-y|^2}{t-s}\right)|b(t,y)|dydt. \qquad (3)$$

Under the above assumption, we proved that the martingale problem for

$$L_t = \frac{1}{2}\triangle + b(t,x) \cdot \nabla$$

is well-posed, or equivalently, the stochastic differential equation (1) has a unique weak solution $(X_t, \mathbf{P}^{s,x})$ for every starting point (s,x).

We should note that $\mathcal{FK}^{\alpha}_{d-1}$ includes the (time-independent) Kato class $\mathcal{K}_{d-1}$, therefore our work extends the results of [3].

5. Transition Density Function Estimates for $(X_t, \mathbf{P}^{s,x})$

When the drift $|b(t,x)|$ is in the forward-Kato class $\mathcal{FK}^{\alpha}_{d-1}$ for some $\alpha < \frac{1}{2}$, from our results, we know that the stochastic differential equation (1) has a unique weak solution $(X_t, \mathbf{P}^{s,x})$. This process $(X_t, \mathbf{P}^{s,x})$ need not have a transition density function.

Now we further assume $|b|$ to be in the time-dependent Kato class $\mathcal{TK}^{\alpha'}_{d-1}$ for some $\alpha' < \frac{1}{4}$, namely

$$\lim_{h\to 0} \left(N_h^{\alpha',+}(|b|) + N_h^{\alpha',-}(|b|)\right) = 0,$$

where

$$N_h^{\alpha',-}(|b|) := \sup_{(t,y)\in[0,\infty)\times\mathbb{R}^d} \int_{t-h}^{t} \int_{\mathbb{R}^d} \frac{1}{(t-s)^{\frac{d+1}{2}}} \exp\left(-\alpha'\frac{|x-y|^2}{t-s}\right)|b(s,x)|dxds$$

and the definition of $N_h^{\alpha',+}(|b|)$ is the same as (3). Then we proved that $(X_t, \mathbf{P}^{s,x})$ has a continuous transition density function $q(s,x;t,y)$ satisfying two sided Gaussian estimates.

References

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