Glauber and Kawasaki dynamics in continuum

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1. Introduction

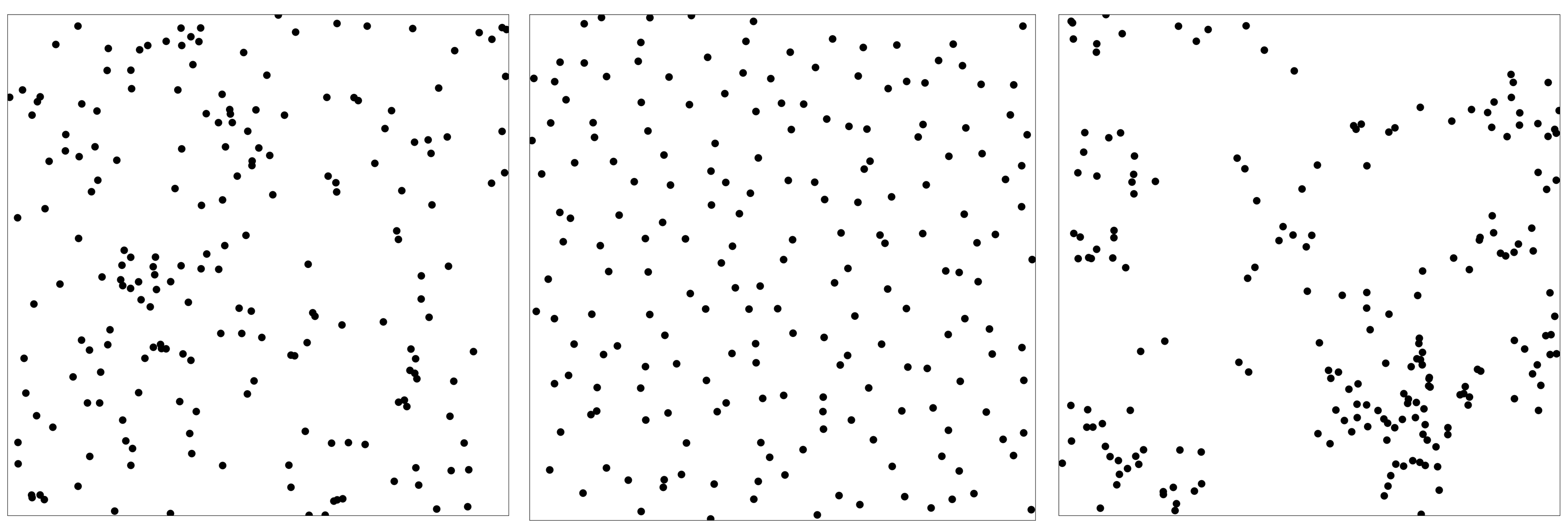
The field of interacting particle systems began as a branch of probability theory in 1960's. The original motivation came from statistical mechanics, later it became clear that the models have many applications in sociology, biology, economics, genetics, epidemology etc. A typical infinite interacting particle system consists of infinitely many particles, interacting in some position space (for example $\mathbb{R}^d$). The behavior of an interacting particle system depends very much on the precise nature of the interaction, and most of the research deals with a certain type of models in which the interaction is of prescribed form. The main problems which have been treated involve the long-time behavior of the system, i.e. the properties as time $t \to \infty$.
In physics, all elementary particles and composite particles are either bosons or fermions (depending on their spin). While fermions obey the Pauli exclusion principle: "no more than one fermion can occupy a single quantum state", there is no exclusion property for bosons, which can occupy the same quantum state.
We consider certain dynamics – birth-and-death, or kind of "hopping particles", which have fermion processes as their invariant measures. The Glauber dynamics is a special type of birth-and-death process in $\mathbb{R}^d$. At every moment of time a particle of our configuration (collection of particles) can die, and at every moment a new particle can be born at any place in $\mathbb{R}^d$, whereas the life time of the particle is exponentially distributed.
In the Kawasaki dynamics interacting particles randomly hop over $\mathbb{R}^d$. It means that at exponentially distributed times a particle of our configuration can disappear, and appear at any other point in $\mathbb{R}^d$.

2. Glauber and Kawasaki dynamics for Determinantal point processes

Fix $\mathbb{R}^d$, $d \in \mathbb{N}$ as position space of paricles, and a Radon, non-atomic measure ν on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$. The configuration space Γ over $\mathbb{R}^d$ is defined as the set of all subsets of $\mathbb{R}^d$ which are locally finite ($|A|$ – number of elements in the set A):

$$\Gamma := \left\{\gamma \subset \mathbb{R}^d : \, |\gamma \cap \Lambda| < \infty \text{ for each compact } \Lambda \subset \mathbb{R}^d\right\}.$$

A point process μ is a probability measure on Γ.
Let K be a linear, Hermitian integral operator on the space $L^2(\mathbb{R}^d, \nu)$ which is locally of trace class and $0 \le K < 1$.
The following samples of translation invariant point processes on the plane (taken from [1]).



From left to right:
Poisson point process – non-interacting particles, free case;
Determinantal point process – strong repulsion properties;
Permanental point process – clumping behavior.
A point process μ is said to have correlation functions if, for any $n \in \mathbb{N}$, there exists a non-negative, measurable, symmetric function $k_\mu^{(n)}$ on $(\mathbb{R}^d)^n$ such that, for any measurable, symmetric function $f^{(n)} : (\mathbb{R}^d)^n \to [0, +\infty]$,

$$\int_\Gamma \sum_{\{x_1,\ldots,x_n\}\subset\gamma} f^{(n)}(x_1, \ldots, x_n)\, \mu(d\gamma) = \frac{1}{n!} \int_{(\mathbb{R}^d)^n} f^{(n)}(x_1, \ldots, x_n) k_\mu^{(n)}(x_1, \ldots, x_n)\, \nu(dx_1) \cdots \nu(dx_n).$$

A point process μ having correlation functions

$$k_\mu^{(n)}(x_1, \ldots, x_n) = \det(K(x_i, x_j))_{i,j=1}^n$$

is called the fermion (or determinantal) point process corresponding to the operator K. If in the definition above we have per instead of det then we get boson (permanental) point process.
We fix a fermion point process μ, and consider corresponding Glauber and Kawasaki dynamics, which have μ as invariant measure. The generator of the Glauber dynamics is given by the formula

$$(H_{\mathrm{G}}F)(\gamma) = \sum_{x\in\gamma} d(x, \gamma)(F(\gamma \setminus x) - F(\gamma)) + \int_{\mathbb{R}^d} b(x, \gamma)(F(\gamma \cup x) - F(\gamma))\, \nu(dx).$$

$d(x, \gamma)$ describes the death rate, while $b(x, \gamma)$ describes the birth rate.
The corresponding Dirichlet form of the Glauber dynamics is given by

$$\mathcal{E}_G(F, F) = \int_\Gamma \sum_{x\in\gamma} d(x, \gamma)(F(\gamma \setminus x) - F(\gamma))^2 \mu(d\gamma).$$

The generator of the Kawasaki dynamics can be written as

$$(H_{\mathrm{K}}F)(\gamma) = 2 \sum_{x\in\gamma} \int_{\mathbb{R}^d} c(x, y, \gamma)(F(\gamma \setminus x \cup y) - F(\gamma))\, \nu(dy),$$

where the coefficient $c(x, y, \gamma)$ describes the intensity of jumps.
Then the Dirichlet form of the Kawasaki dynamics is

$$\mathcal{E}_K(F, F) = \int_\Gamma \int_{\mathbb{R}^d} \sum_{x\in\gamma} c(x, y, \gamma \setminus x)(F(\gamma \setminus x \cup y) - F(\gamma))^2 \nu(dy)\mu(d\gamma).$$

Using the theory of Dirichlet forms [2], we prove existence of both dynamics which have fermion point processes as symmetrizing measures.

3. Spectral gap of Glauber dynamics generator

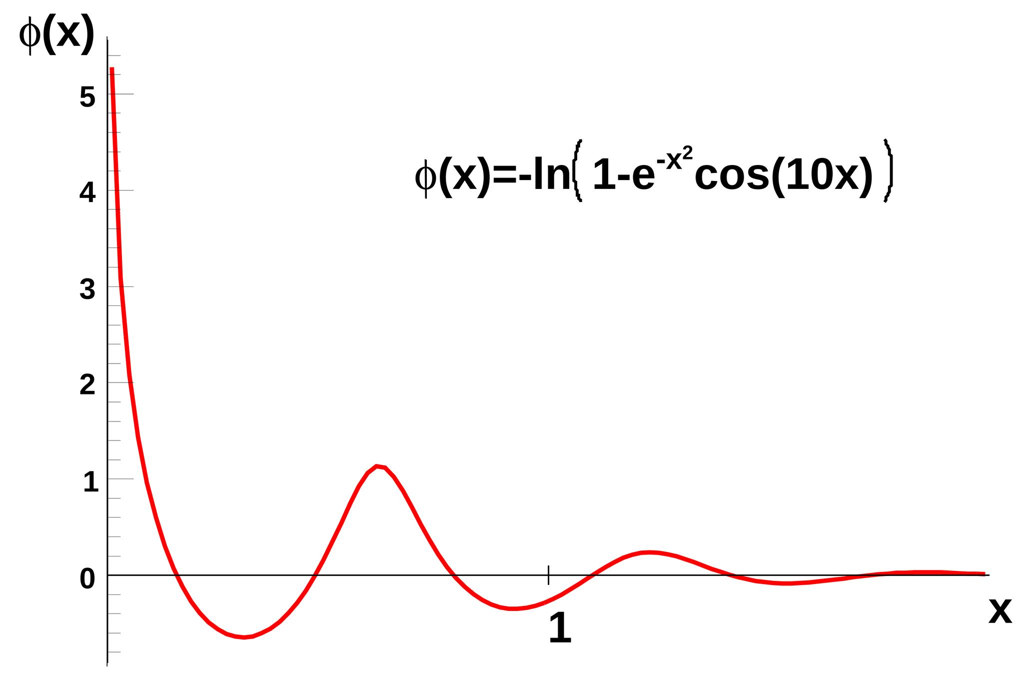
Another important question which arises in connection with different dynamics is the rate of convergence to equilibrium. One of the characteristics which give us the information about the speed of convergence is the spectral gap of the generator. If the spectral gap exists, then convergence is of exponential rate. For the Glauber dynamics in continuum with invariant Gibbs measure the problem of the existence of the spectral gap was studied in a whole range of articles, In all of them the existence of spectral gap was obtained by using different methods. Nevertheless, in all cases the potential is assumed to be positive, and this assumption is crucial for the proof. Therefore there emerged a question, whether the spectral gap exists in the case when the potential has a negative part. We can show the existence of the spectral gap for a certain class of pair potentials, which do not have to be positive.
We consider the Glauber dynamics on $\mathbb{R}^d$ with $d \equiv 1$ and invariant Gibbs measure μ. A Gibbs measure μ, corresponding to a translation invariant pair potential ϕ and activity $z > 0$ is given heuristically through

$$\mu = \frac{1}{Z} \exp\left\{-\sum_{x,y\in\gamma} \phi(x - y)\right\} d\pi_z,$$

where $\phi : \mathbb{R}^d \to (-\infty, \infty]$ is the pair potential – symmetric function, π_z – Poisson measure.
Consider the class $\mathcal{K}$ of pair potentials ϕ of the form

$$\phi := -\ln(1 - f),$$

where f is a continuous positive definite function such that $f(0) \le 1$.
For potentials from the class $\mathcal{K}$ the corresponding Gibbs measure μ exists, and the generator H fulfills the coercivity inequality for $c = 1$. The class $\mathcal{K}$ contains also non-positive potentials, for example the one in the picture below.



References

[1] J. Ben Hough, M. Krishnapur, Y. Peres and B. Virg, Determinantal Processes and Independence, Probability Surveys, 3, (2006), 206–229.

[2] Z.-M. Ma, M. Röckner, An Introduction to the Theory of (Non-Symmetric) Dirichlet Forms, Springer-Verlag, 1992.