Harnack Inequalities and Applications for Stochastic Equations

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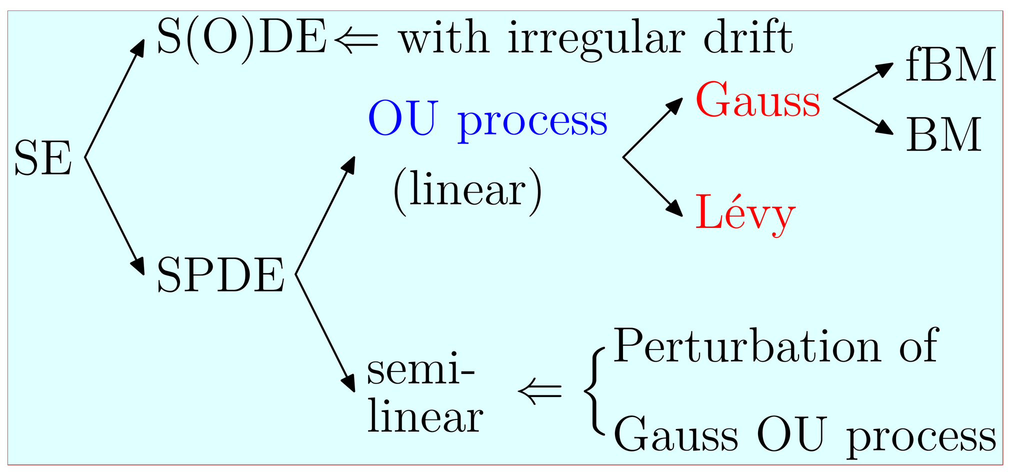
1. Introduction

1.1 Stochastic Equations

We consider Harnack inequalities and their applications for the following stochastic equations (SEs).

Single-valued SEs: $dX_t = AX_t\,dt + F(t, X_t)\,dt + dL_t$.



Multi-valued SEs: $dX_t \in \tilde{A}X_t\,dt + BX_t\,dt + \sigma_t dW_t$.

MSDEs (on $\mathbb{R}^d$) & MSEEs (on Gelfand triple)

We just show the main technical by establishing Harnack inequalities for OU processes.

1.2 Transition Semigroup

Let $\mathbb{H}$ be a real separable Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and norm $|\cdot|$. Let X be a solution process to some stochastic equation.

For every $t \in [0, \infty)$, $X_t(\cdot)$, $X_t\colon (\Omega, \mathscr{F}_t, \mathbb{P}) \to (\mathbb{H}, \mathcal{B}(\mathbb{H}))$ is a r.v.. Denote the marginal distribution of X_t by μ_t^x:

$$\mu_t^x := \mathbb{P} \circ (X_t^x)^{-1}.$$

Then we define the transition semigroup by

$$P_t f(x) = \int_\Omega f(X_t^x)\,d\mathbb{P} = \mathbb{E}_{\mathbb{P}} f(X_t^x) = \int_{\mathbb{H}} f(z)\,d\mu_t^x$$

for every bounded measurable function f on $\mathbb{H}$.

1.3 Harnack Inequalities

The Harnack inequality we are interested in which is first introduced by Wang in 1997 [2], is of the following form

$$(P_t f)^\alpha(x) \le C(t, \alpha, x, y) P_t f^\alpha(y),$$

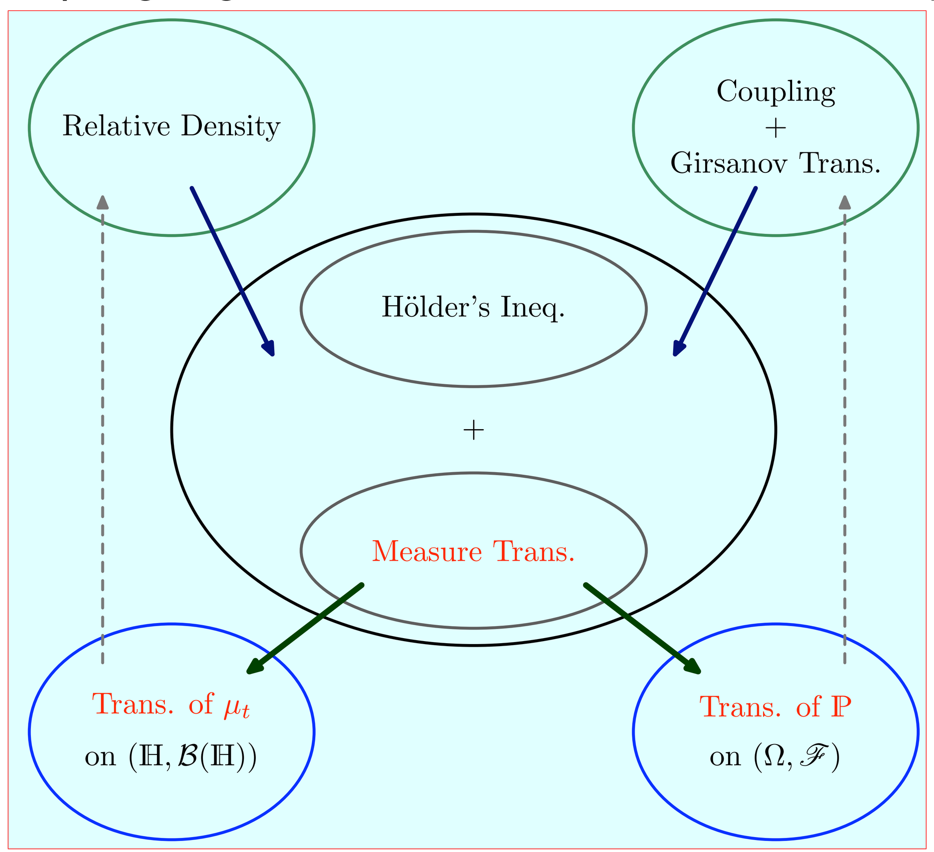
for every $x, y \in \mathbb{H}, f \in \mathscr{C}_b^+(\mathbb{H}), \alpha > 1$, where $C(t, \alpha, x, y)$ is a constant independent of f.

Applications of Harnack inequalities including Regularizing Property, Contractivity, Heat Kernel Estimates, Bound of the Norm of Heat Kernel etc..

2. Main Technique

2.1 How to get Harnack inequalities?

The main technique we use is applying Hölder's inequality after a measure transformation. There are two levels of measure transformation which are used in [3] (Relative density method) and [1] (Coupling argument+Girsanov's transformation).



We use the image measure transformation for Gauss case; and use the measure transformation on Ω for Lévy case.

2.2 Preliminary: Null Controllable

Consider the system

$$\begin{cases} dx_t = Ax_t\,dt + R^{1/2}u_t\,dt, \\ x_0 = y - x \in \mathbb{H}. \end{cases} \qquad (1)$$

We say a system of the form (1) is Null Controllable in time T if for each initial data, there exist $u \in L^2([0,T], \mathbb{H})$ such that $x_T = 0$.

Define $Q_T = \int_0^T S_t R S_t^*\,dt$. Then

$$\text{Null Controllable} \Leftrightarrow S_T(\mathbb{H}) \subset Q_T^{1/2}(\mathbb{H})$$

Let $\Gamma_T = Q_T^{-1/2} S_T$. We have a representation for the Minimal Energy of driving the initial states $y - x$ to 0 in time T:

$$|\Gamma_T(x-y)|^2 = \inf\left\{\int_0^T |u_s|^2\,ds \colon u \in L^2\right\}.$$

3. Gauss Case

3.1 Gauss OU Process and Semigroup

Mild solution of the Langevin's Equation:

$$dX_t = AX_t\,dt + R^{1/2}\,dW_t, \quad X_0 = x \in \mathbb{H}.$$

Assume that Q_t, $0 \le t \le \infty$, is of trace class. Then

$$X_t = S_t x + \int_0^t S_{t-s} R^{1/2}\,dW_s, \quad t \ge 0.$$

We know

$$X_t \sim N(S_t x, Q_t).$$

Therefore the Gaussian Ornstein-Uhlenbeck Semigroup is given by

$$P_t f(x) = \int_{\mathbb{H}} f(S_t x + z)\,N(0, Q_t)(dz).$$

3.2 Harnack Inequality for Gaussian OU Semigroup

Assume

- Q_t is of trace class;
- $S_t(\mathbb{H}) \subset Q_t^{1/2}(\mathbb{H})$.

By using the Cameron-Martin formula for Gaussian measures, we have

$$(P_t f)^\alpha(x) \le \exp\left(\frac{\beta}{2}|\Gamma_t(x-y)|^2\right) \cdot (P_t f^\alpha(y)), \qquad (2)$$

for every $t > 0$, $f \in \mathscr{C}_b^+(\mathbb{H})$, $x, y \in \mathbb{H}$ and $\alpha, \beta > 1$ satisfying $\frac{1}{\alpha} + \frac{1}{\beta} = 1$.

3.3 Application: Strong Feller Property

For Gauss OU semigroup P_t. The following statements are equivalent to each other.

- $S_t(\mathbb{H}) \subset Q_t^{1/2}(\mathbb{H})$;
- Harnack inequality holds;
- P_t is strongly Feller.

4. Lévy Case

4.1 Transformation of Measures on Ω

Let $(X_t^x, \mathbb{Q})$ and $(Y_t^y, \mathbb{P})$ be two processes on $(\Omega, \mathscr{F}_T)$. Assume three conditions

1. $P_t f(x) = \mathbb{E}_{\mathbb{Q}} f(X_t^x)$, $P_t f(y) = \mathbb{E}_{\mathbb{P}} f(Y_t^y)$;
2. $\mathbb{Q} = \rho_T \mathbb{P}$;
3. $X_T^x = Y_T^y$.

Then

$$\begin{aligned} P_T f(x) &= \mathbb{E}_{\mathbb{Q}} f(X_T^x) = \mathbb{E}_{\mathbb{Q}} f(Y_T^y) = \mathbb{E}_{\mathbb{P}} \rho_T f(Y_T^y) \\ &\le (\mathbb{E}_{\mathbb{P}} \rho_T^\beta)^{1/\beta} (\mathbb{E}_{\mathbb{P}} f^\alpha(Y_T^y))^{1/\alpha} \\ &= (\mathbb{E}_{\mathbb{P}} \rho_T^\beta)^{1/\beta} (P_T f^\alpha(y))^{1/\alpha}. \end{aligned}$$

Hence we get the following Harnack inequality

$$(P_T f)^\alpha(x) \le [\mathbb{E}_{\mathbb{P}} \rho_T^\beta]^{\frac{\alpha}{\beta}} P_T f^\alpha(y).$$

4.2 Girsanov's Theorem for Lévy Process

Let $(Z_t)_{0 \le t \le T}$ be a Lévy process with characteristic triple (b, R, ν) under probability measure $(\Omega, (\mathscr{F}_t)_{0 \le t \le T}, \mathbb{P})$. Under some new probability measure $\mathbb{Q} = \rho^W(T)\mathbb{P}$,

$$\tilde{Z}(t) := Z(t) - \int_0^t \psi(s)\,ds, \quad 0 \le t \le T$$

is also a (b, R, ν)–Lévy process on $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{0 \le t \le T})$, where we will take $\psi(t) = R^{1/2} u_t$, $0 \le t \le T$, and u_t is the null control of the system (1).

4.3 Lévy Driven OU Process

Consider Y_t^y driven by Z_t on $(\Omega, \mathscr{F}_t, \mathbb{P})$:

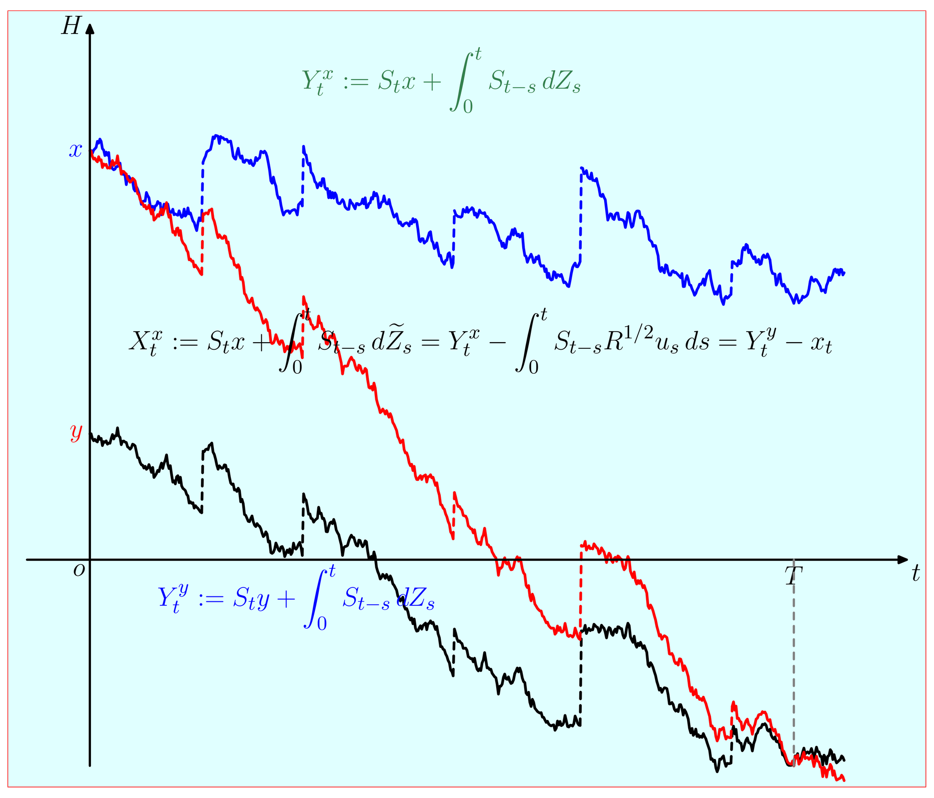
$$\begin{aligned} Y_0^y &= y, \\ dY_t^y &= AY_t^y\,dt + dZ_t. \end{aligned}$$

We define $P_t f(y) = \mathbb{E}_{\mathbb{P}} f(Y_t^y)$. We also consider X_t^x driven by $\tilde{Z}_t$ on $(\Omega, \mathscr{F}_t, \mathbb{Q})$. ($\mathbb{Q} = \rho^W \mathbb{P}$)

$$\begin{aligned} X_0^x &= x, \\ dX_t^x &= AX_t^x\,dt + d\tilde{Z}_t, \\ &= AX_t^x\,dt + dZ_t - R^{1/2} u_t\,dt. \end{aligned}$$

We see $P_t f(x) = \mathbb{E}_{\mathbb{Q}} f(X_t^x)$. By noting that $\tilde{Z}_t$ is a drift transformation of Z_t and note that $x_T = 0$, we get

$$X_t^x = Y_t^y - x_t \Rightarrow X_T = Y_T$$



4.4 Harnack Inequality for Lévy Driven OU Process

By the procedure introduced in subsection 4.1, for Lévy OU semigroup P_t, we have

$$(P_t f)^\alpha(x) \le \exp\left(\frac{\beta}{2}\int_0^t |u_s|^2\,ds\right) \cdot (P_t f^\alpha(y)),$$

$\forall t > 0, \alpha, \beta > 1$ satisfying $\frac{1}{\alpha} + \frac{1}{\beta} = 1$, $f \in \mathscr{C}_b^+(\mathbb{H})$, $x, y \in \mathbb{H}$.

Taking infimum over all null control u, we still can get an inequality with the same form as (2).

References

[1] M. Arnaudon, A. Thalmaier, and F.-Y. Wang, *Harnack inequality and heat kernel estimates on manifolds with curvature unbounded below*, Bull. Sci. Math. **130** (2006), no. 3, 223–233.

[2] F.-Y. Wang, *Logarithmic Sobolev inequalities on noncompact Riemannian manifolds*, Probab. Theory Related Fields **109** (1997), no. 3, 417–424.

[3] M. Röckner and F.-Y. Wang, *Harnack and functional inequalities for generalized Mehler semigroups*, J. Funct. Anal. **203** (2003), no. 1, 237–261.