1. Introduction

1.1 Stochastic Equations
We consider Harnack inequalities and their applications for the following stochastic equations (SEs).

- Single-valued SEs: \( dX_t = AX_t dt + f(t, X_t) + RdtdW_t \) with irregular drift.\( \Rightarrow \) BM
- Gaussian OU process
- Perturbation of Gaussian OU process

- Multi-valued SEs: \( dX_t \in AX_t dt + B(X_t) dt + \sigma_dW_t \) with singular drift.\( \Rightarrow \) BM
- MSDEs (on \( \mathbb{R}^d \)) \& MSEEs (on Gelfand triple)

We just show the main technique by establishing Harnack inequalities for OU processes.

1.2 Transition Semigroup

Let \( \{ P_t \} \) be a real separable Hilbert space with inner product \( \langle \cdot, \cdot \rangle \) and norm \( | \cdot | \). Let \( \lambda \) be a solution process to some stochastic equation:

For every \( t \in [0, \infty) \), \( X_t \), \( (t, x, \mathcal{P}) = (H, \mathcal{B}(H)) \) is a r.v. Denote the marginal distribution of \( X_t \) by \( \mu_t \):

\[ \mu_t = P_t \circ \{ X_t \} \]

Then we define the transition semigroup by

\[ P_t f(x) = \int f(y) d\mu_t(y) \]

for every bounded measurable function \( f \) on \( H \).

1.3 Harnack Inequalities

The Harnack inequality we are interested in is first introduced by Wang in 1997 [2], of the following form

\[ |P_t f| H \leq C(t, a, x, y, g) |P_t f| H \]

for every \( x, y \in H, f \in C^\infty_0(H), a > 1 \), where \( C(t, a, x, y) \) is a constant independent of \( f \).

Applications of Harnack inequalities including Regularizing Property, Contradictivity, Heat Kernel Estimates, Bound of the Norm of Heat Kernel etc.,

2. Main Technique

2.1 How to get Harnack inequalities?

The main technique we use is applying Hölder’s inequality after a measure transformation. There are two levels of measure transformation which are used in [3] (Relative density method) and [1] (Coupling argument+Girsanov’s transformation).

We use the image measure transformation for Gauss case; and use the measure transformation on \( \Omega \) for Lévy case.

2.2 Preliminary: Null Controllable

Consider the system

\[ \begin{align*}
  dx_t &= Ax_t dt + R^{1/2}u_t dt, \\
  x_0 &\in H,
\end{align*} \]

We say a system of the form (1) is Null Controllable in time 7 if for each initial data, there exist \( u \in L^2(0, T; H) \) such that \( x_2 = 0 \).

Define \( Q_t = \mathbb{E} f(X_t) \) dt. Then

Null Controllable \( \Rightarrow S(H) \subset Q^{1/2}(H) \)

Let \( Y_t = Q^{1/2}S_{t\alpha} \). We have a representation for the Minimal Energy \( E \) of driving the initial states \( y \) to 0 in time 7:

\[ E_{\min}(y - x, 0) = \inf \left\{ \int_0^T |u(t)|^2 dt : u \in L^2(0, T; H) \right\} \]

3. Gauss Case

3.1 Gauss OU Process and Semigroup

Mild solution of the Langevin’s Equation:

\[ \begin{align*}
  dx_t &= AX_t dt + R^{1/2}dW_t, \\
  x_0 &\in H, \ 
\end{align*} \]

Assume that \( Q_{\alpha} \leq T \leq \infty \), is of trace class then

\[ X_t = S_t x + \int_0^T S_{t-s} R^{1/2} dW_s, \quad t \geq 0. \]

We know

\[ \begin{align*}
  X_t &\sim N(S_t x, Q_{\alpha}^t) \\
  \text{Therefore the Gaussian Ornstein-Uhlenbeck Semigroup is given by} \\
  P_t f(x) &= \int f(S_t x + z) N(0, Q_{\alpha}^t)(dz).
\end{align*} \]

3.2 Harnack Inequality for Gaussian OU Semigroup

Assume

\[ \begin{align*}
  &\cdot Q_{\alpha} \text{ is of trace class;} \\
  &\cdot S(H) \subset Q^{1/2}(H).
\end{align*} \]

By using the Cameron-Martin formula for Gaussian measures, we have

\[ |P_t f(x)| H \leq \exp \left( -\frac{1}{2} \langle y - x, \gamma \rangle \right) \cdot (P_t f(y)) \]

for every \( t > 0, f \in C^\infty_0(H), x, y \in H \) and \( \alpha, \beta > 1 \) satisfying \( 2 \alpha \beta > 1 \).

3.3 Application: Strong Feller Property

For Gauss OU semigroup \( P_t \). The following statements are equivalent to each other:

\[ \cdot S(H) \subset Q^{1/2}(H); \]

\[ \cdot \text{Harnack inequality holds;} \]

\[ \cdot P_t \text{ is strongly Feller.} \]

4. Lévy Case

4.1 Transformation of Measures on \( \Omega \)

Let \( (X_t, Y_t) \) and \( Y_t \) be two processes on \( (\Omega, \mathcal{F}, \mathbb{P}) \). Define three conditions:

1. \( P_1(f) = \mathbb{E}(f(X_t)) = \mathbb{E}(f(Y_t)) \)
2. \( Q_1 = \mathbb{P} \)
3. \( Y_0 = Y_1 \)

Then

\[ \begin{align*}
  P_t f(x) &= \mathbb{E}(P_t f(X_t)) = \mathbb{E}(P_t f(Y_t)) \leq \mathbb{E}((P_t f(Y_t))^{1/\alpha}) \leq \mathbb{E}(P_t f(Y_t))^{1/\alpha}.
\end{align*} \]

4.2 Girsanov’s Theorem for Levy Process

Let \( (Z_t)_{t \geq 0} \) be a Lévy process with characteristic triple \((\xi, \nu, \gamma)\) under probability measure \((\Omega, \mathcal{F}, \mathbb{P})\). Under some new probability measure \( \mathbb{P}^{\gamma_{t\alpha}} \), \( Z(t) = Z(0) - \xi \cdot \mathbb{E}(\gamma_{t\alpha} | Z(0)) \), \( Z(t) = Z(0) - \gamma_{t\alpha} \), \( 0 \leq t \leq T \), and \( \gamma_{t\alpha} \) is the null control of the system (1).

4.3 Lévy Driven OU Process

Consider \( Y_t \) driven by \( Z_t \) on \( (\Omega, \mathcal{F}, \mathbb{P}) \):

\[ Y_t = \gamma_{t\alpha} \]

Define \( P_t \mathbb{P} = \mathbb{P}^{\gamma_{t\alpha}} \). We consider \( X_t \) driven by \( Z_t \) on \( (\Omega_t, \mathcal{F}_t, \mathbb{Q}) \):

\[ X_t = Z_t \]

We see \( P_t \mathbb{P} = \mathbb{P}^{\gamma_{t\alpha}} \). By noting that \( \tilde{Z} \) is a drift transformation of \( Z \), and note that \( x_0 = 0 \), we get

\[ \mathbb{E}(f(X_t)) \leq \mathbb{E}(f(Y_t)) \]

4.4 Harnack Inequality for Lévy Driven OU Process

By the procedure introduced in subsection 4.1 for Lévy OU semigroup \( P_t \), we have

\[ |P_t f(x)| H \leq \exp \left( -\frac{1}{2} \left\langle x, \gamma \right\rangle \right) \cdot (P_t f(y)) \]

\[ \forall t > 0, \alpha \beta > 1 \text{ satisfying } \frac{1}{\alpha} + \frac{1}{\beta} = 1, f \in C^\infty_0(H), \ x, y \in H. \]

Taking infimum over all null control, we can still get an inequality with the same form as (2).

References

