SCENERY RECONSTRUCTION

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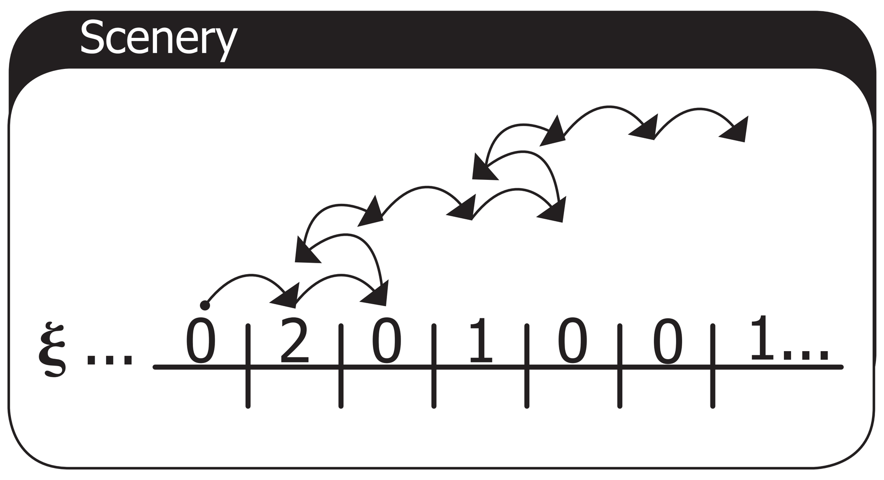
1. Introduction

The basic scenery reconstruction problem can be described as follows: suppose that each $z \in \mathbb{Z}$ was randomly colored, this coloration will be called a scenery on $\mathbb{Z}$ and denoted by ξ. Then, a simple random walk $\{S(t)\}_{t\in\mathbb{N}}$ starts to move on these colored integers ξ registering the color it sees at each time t, thus producing a new sequence of colors $\{\chi(t)\}_{t\in\mathbb{N}}$. We illustrate this with the next example.



So, following the path above on ξ we obtain the sequence of observations χ

0 2 0 2 0 1 0 1 0 0 1

The interesting question is: Can the original coloring of the integer numbers ξ be reconstructed (possibly up to shift and/or reflection) with that sequence $\{\chi(t)\}_{t\in\mathbb{N}}$ produced by the process? The answer in general is NO. However, under appropriate restrictions, the answer will become YES. Let us explain these restrictions:

1. Two sceneries are called equivalent if one of them is obtained from the other by a translation or reflection. If ξ and ξ^* are equivalent, we can in general not distinguish whether the observations come from ξ or ξ^*. Thus, we can only reconstruct ξ up to equivalence.

2. The reconstruction works in the best case only almost surely.

2. The evolution of scenery reconstruction

The development of the theory of scenery reconstruction took place in three phases. In each phase, it became possible to reconstruct scenery in a more complicated setting.

1. Combinatorial case: Sceneries with two or more colors along a simple random walk path

2. Semi-combinatorial case: 2-color scenery along a simple random walk with holding

3. Purely statistical case: 2-color scenery along a random walk with jumps

There are other cases, which are strongly different from the three above:

1. Scenery reconstruction given disturbed input data

2. Scenery reconstruction in two dimensions

3. Our problem

In [2], Matzinger showed how to reconstruct a 3-color scenery seen along a random walk path. For this however he needs an infinite amount of observations, hence the algorithm is not practical. In [3], Matzinger and Rolles prove that in certain cases finite pieces of sceneries can be reconstructed close to the origin. The method they describe is almost impossible to implement, hence it is a more theoretical result.

We work the problem that a finite piece of scenery can be reconstructed in polynomial time with high probability. That means that we only have finitely many observations. The number of observations we are allowed to see is polynomial in the length of the piece we reconstruct. We also try to implement such a reconstruction algorithm in practice. All this is done in the context of a 3-color scenery seen along a simple random walk.

4. Main ideas

1. Substrings.

Let S be a string from ξ of length l. Assume that we have two disjoint intervals in the string both of length $K \ln l$. We obtain that:

- in an i.i.d. 3-equiprobable-color string of length order l, every substring of length $K \ln l$ appears only in one location with high probability, (assuming that K is a large enough constant).

So, we are going to reconstruct a string S using substrings.

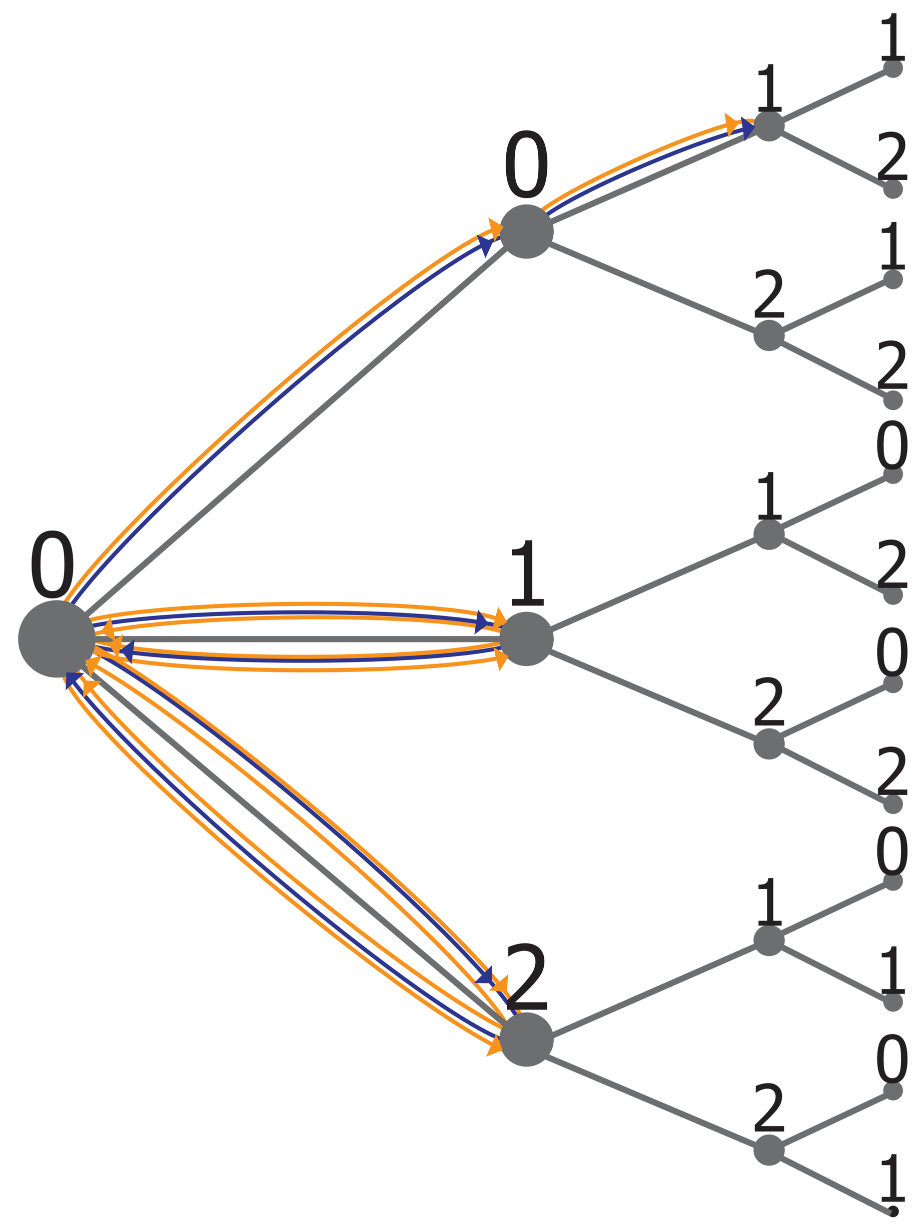
2. Shortest time.

Assume that x and y are two non-random integer numbers such that $x < y$. Assume that we want to reconstruct the "word" written between x and y, that is we would like to reconstruct $\xi_x\xi_{x+1}\ldots\xi_y$. For this assume that you are given the observations $\chi_1\chi_2\ldots$. Then, when the random walk crosses in shortest time from x to y, we have that the random walk takes only steps to the right. Hence during such a minimal time crossing we have that in the observations we see a copy of the word $\xi_x\xi_{x+1}\ldots\xi_y$.

Since the random walk is recurrent, then, we have that it will cross in shortest time from x to y infinitely often.

3. representation of ξ over a 3-regular tree.

Let (E, V) be a 3-regular tree and ψ a coloring on it such that the color of the origin $\psi(v_0) = \xi_0 = \chi_0$ which is known, furthermore for every vertex $v \in V$ the color from its 3-adjacent vertices corresponding to $\{0, 1, 2\}$. We use a combinatorial approach representing the scenery ξ as a nearest neighbor walk R on a 3-regular tree, then, we try to reconstruct R using the observations χ. How? Representing χ itself as a nearest neighbor walk on the tree, it is just $R \circ S$. Hence, the color record χ uniquely determines $R \circ S$, thus, the problem of reconstructing ξ is equivalent to reconstructing R given $R \circ S$. The next figure shows the representation for the example given at the begining. The blue path is just the scenery ξ, while the orange path is the sequence of observations χ.



We construct an algorithm that tries to reconstruct a piece of ξ which contains the string $\xi(-n)\xi(-n+1)\xi(-n+2)\ldots\xi(n)$ and is contained in the string $\xi(-4n)\xi(-4n+1)\xi(-4n+2)\ldots\xi(4n)$, before a time $T = n^{2(b+1)}$, with $b > K \ln 2$.

We are using combinatorics methods in order to obtain a proof for the event that the reconstruction algorithm works, holds with hight probability.

References

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[4] Lindenstrauss (1999) Indistinguishable sceneries. *Random Structures Algorithms.* **14** 71–86.