Continuous Contact Model with Jumps – A Plankton Dynamics Model –

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1. Summary

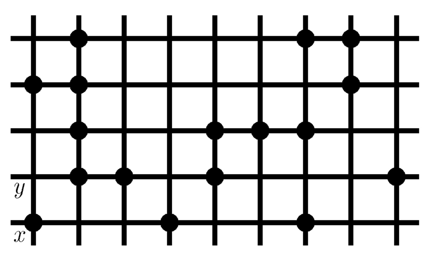
The contact model is well-known in lattice theory. In recent years, a continuous version of this model has been developed, cf. [2].
In [4], we modified the continuous model by adding a jump-part to the generator describing the model. Thus one obtains a birth-and-death model, where the individuals are allowed to move in space.
We constructed a corresponding process and proved existence of invariant measures in dimension $d \geq 2$ in the critical, translation-invariant case under certain conditions on the dispersion and the jump distributions.

2. Lattice Contact Model

Lattice models are well-studied examples of interacting particle systems and of individual based models, cf. e.g. [1]. Famous models in this area appear in statistical physics, e.g. lattice gas models. But lattice models also find applications in many other disciplines, e.g. to describe interaction of agents or individuals in economy or sociology and population biology, resp.
A special class of population models are the so-called **birth-and-death processes**. Individuals can appear on free lattice sites ("birth") or disappear ("death") by certain random rules depending on their environment (e.g. offspring or death because of stress in overcrowded areas).
Mathematically, this is modelled in the following way: $\mathbb{Z}^d$ is endued with the usual graph structure, i.e. two points $x, y \in \mathbb{Z}^d$ are neighbors iff their distance from each other is exactly 1. Then one assigns a "spin" σ to each site x of the lattice:

$$\sigma(x) = \begin{cases} 1, & \text{if } x \text{ is occupied by an individual,} \\ 0, & \text{if } x \text{ is free.} \end{cases}$$

This way, one obtains a **(lattice) configuration** $\sigma = (\sigma(x))_{x\in\mathbb{Z}^d}$. In the picture ($d = 2$), occupied sites are represented by a dot:



$\sigma(x) = 1$, $\sigma(y) = 0$

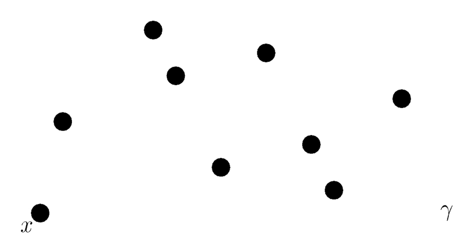
Then the dynamics of the system is described by a stochastic process on the space of all lattice configurations.
A special birth-and-death process is the **contact model**. (The name arises from modelling of infection spreading.) In this model, individuals randomly (independently of each other) give birth to new ones appearing on free neighbor lattice sites. And each individual dies after a random time with fixed rate.

3. Configuration Spaces

In recent years, **continuous versions of many models** from lattice theory have been studied. Here, the particles (or individuals, agents, ...) are not bound to the sites of a lattice $\mathbb{Z}^d$ but live in the complete space $\mathbb{R}^d$ or on a (Riemannian) manifold.
The mathematical framework for such models are **(continuous) configuration spaces**. The configuration space $\Gamma(\mathbb{R}^d)$ consists of all locally finite subsets, i.e.

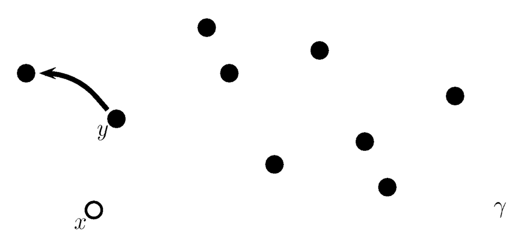
$$\Gamma(\mathbb{R}^d) := \{\gamma \subset \mathbb{R}^d : |\gamma \cap K| < \infty \text{ for any compact } K \subset \mathbb{R}^d\}. \quad (1)$$

I.e., one considers infinitely many (indistinguishable) particles in $\mathbb{R}^d$, but only finitely many in any bounded ball.



4. Continuous Contact Model

Also a **continuous contact model** has been developed, cf. [2]. Since $\mathbb{R}^d$ does not possess a graph structure, one does not have the notion of neighbors. In lieu thereof an individual at point $x \in \mathbb{R}^d$ can give birth to a new one at any other point $y \in \mathbb{R}^d$. This is controlled by a **dispersion kernel** $a : \mathbb{R}^d \to \mathbb{R}_+$. Here a is a probability density function, i.e. a non-negative function with $\int a\,dx = 1$. As in the lattice case, individuals die after a random time with constant rate 1.



x dies, y gives birth

This dynamics can be described by the following **(formal) generator** on $\Gamma(\mathbb{R}^d)$:

$$L_C F(\gamma) = \sum_{x\in\gamma}\Big(F(\gamma\setminus x) - F(\gamma)\Big) + \varkappa \sum_{y\in\gamma}\int_{\mathbb{R}^d} a(x-y)\Big(F(\gamma\cup x) - F(\gamma)\Big)\,dx. \quad (2)$$

Here, $\varkappa > 0$ is a parameter describing the rate of producing offspring. Corresponding stochastic processes have been constructed for different functions a (with finite or infinite range, see [2, 3]). An interesting question is how the **density** ρ_t of individuals in the space evolves in time depending on the dispersion rate $\varkappa$. (In mathematical/physical terms this is the time evolution of the first correlation function.) In [3] it is shown that

$$\rho_t \begin{cases} \to 0, & \varkappa < 1 \quad \text{sub-critical,} \\ \to \infty, & \varkappa > 1 \quad \text{super-critical,} \\ = \text{const}, & \varkappa = 1 \quad \text{critical.} \end{cases} \quad (3)$$

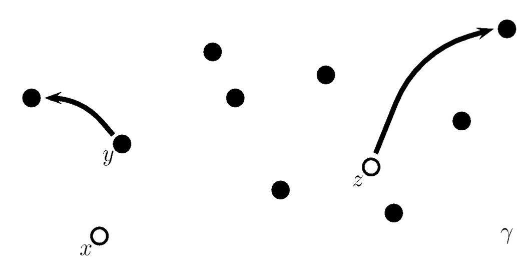
That means, that the population dies out or explodes if the dispersion rate is smaller or higher than the death rate, resp.

5. Continuous Contact Model with Jumps

We have studied a modification of the continuous contact model, namely a **contact model with jumps**. In the classical contact model, one considers only a birth-and-death dynamics. The individuals are born and die, but their position is fixed. (Think of plants.) We have added **motion of the individuals**, namely, we allow them to perform jumps. The corresponding **(formal) generator** has the following form:

$$L_{CJ}F(\gamma) = L_C F(\gamma) + \sum_{x\in\gamma}\int_{\mathbb{R}^d} w(x-y)\Big(F(\gamma\setminus x\cup y) - F(\gamma)\Big)\,dy. \quad (4)$$

The extra term is the generator of a so-called **(free) Kawasaki dynamics**. It describes the jump of an individual from $x \in \mathbb{R}^d$ to the point $y \in \mathbb{R}^d$. w controls the length of jumps. It is an integrable non-negative even function, but not necessarily a probability density, i.e., not necessarily $\int w = 1$.



x dies, y produces offspring, z jumps

The construction of the process works similarly as for the usual contact model. It turns out, that also the evolution of the density is the same as before, i.e., (3) is still valid.
But mathematically, this model has better properties than the usual contact model. It is an interesting problem to show the existence of an **invariant measure** for the dynamics. In [3], the authors give criteria which ensure invariant measures for the contact model in the critical, translation-invariant case, but only in dimension $d \geq 3$. The interesting case $d = 2$ does not work. Our main mathematical motivation to study the contact model with jumps was the idea that jumps might help to overcome this problem, and indeed we could develop a criterion on w (namely an integrability condition for its Fourier transform) to obtain the corresponding result also for $d = 2$.

6. Application: Plankton Dynamics

The continuous contact model with jumps can serve as a model of **plankton dynamics**. The contact model part describes birth-and-death of individuals, and the jump part models the mechanical motion.
In the literature (see e.g. [5]), the motion of plankton is often modelled by a continuous process, i.e. diffusion, since individuals usually do not jump. But birth-and-death and motion of individuals happen on **different time scales**, e.g. motion in terms of minutes and birth-and-death in terms of days. So, if one considers the motion in the birth-and-death time scale, it is appropriate to model it as jumps.

7. Further results

In [4], two further models are discussed, namely **invariance principles** for a diffusion in a random environment and a tagged particle process. For the diffusion, the environment process is constructed via a Dirichlet forms method. In the second model, this has been done before. In both cases, a general method is used to obtain invariance principles. For details and references, see [4].

References

[1] T.M. Liggett, "Interacting Particle Systems", Springer, 1985.
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