Interacting particle systems on graphs

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1. Introduction

This thesis deals with the theory of *Gibbs measures*, which is a branch of classical statistical physics and also a part of probability theory. The notion of a Gibbs measure dates back to the pioneering papers of R. L. Dobrushin [Do 1968, Do 1970] and O. E. Lanford and D. Ruelle [LaRu 1969, Ru 1969]. Gibbs measures describe equilibrium states of a physical system which consists of a very large number of interacting particles. Physically, we attempt to explain the macroscopic behavior of matter on the basis of its microscopic structure. The underlying structure in the whole manuscript is a graph $G(\mathbb{V},\mathbb{E})$. For this case we obtain a series of results concerning existence, uniqueness and non-uniqueness (phase transition) of the corresponding Gibbs measures.

2. Construction on infinite graphs

The construction of the model is the following. We look at an interacting system with a large number of particles. We start with a set $\mathbb{V}$ which labels the particles of the system. The possible states of each particle are described by the set $\mathbb{R}^\nu$, which will give rise to some difficulties since we are in the *non-compact* spin space situation. Having specified these two sets, we can describe a particular state of the total system by a suitable element $\sigma := (\sigma(x))_{x\in\mathbb{V}}$ of the product space

$$\Omega := [\mathbb{R}^\nu]^{\mathbb{V}} := \{\sigma = (\sigma(x))_{x\in\mathbb{V}} \mid \sigma : \mathbb{V} \to \mathbb{R}^\nu\}.$$

Ω is called the *configuration space*. In mathematical language, we consider an interacting system of spins living on a graph $G(\mathbb{V},\mathbb{E})$. Each $x \in \mathbb{V}$ corresponds with variables $\sigma(x)$ called *spins* which take values in $\mathbb{R}^\nu$. We want to describe these systems by a probability measure μ on Ω which has the form in equation (2) below. In order to do so we specify a so-called Hamiltonian $E : \Omega \to \mathbb{R}$ which assigns to each configuration $\sigma \in \Omega$ its potential energy

$$E(\sigma) := \sum_{\substack{x,y\in\mathbb{V}\\ x\sim y}} W_{xy}(\sigma(x),\sigma(y)) + \sum_{x\in V} U_x(\sigma(x)), \quad (1)$$

where $W_{xy}(\sigma(x),\sigma(y))$ is the *pair interaction potential* and $U_x(\sigma(x))$ the *self-interaction potential*. Both are real valued. The equilibrium state of these systems with Hamiltonian E is described by the probability measure

$$\mu(d\sigma(x)) = \frac{1}{Z}e^{-\beta E(\sigma)} \prod_{x\in\mathbb{V}} d\sigma(x) \quad (2)$$

on Ω. The notation $d\sigma(x)$ refers to the Lebesgue measure on $\mathbb{R}^\nu$, $\beta := \frac{1}{k_B T} > 0$ is the *inverse* (absolute) *temperature* with k_B denoting the Boltzmann constant and $Z > 0$ is a *normalizing constant*. The measures μ are called *Gibbs measures*. One of the main question we deal with is: how can we ensure that such a measure really exists on an infinite graph $G(\mathbb{V},\mathbb{E})$? The other main questions are: on which conditions do we have uniqueness or non-uniqueness?

3. Existence of Gibbs measures

Let us consider, for a moment, graphs which have *globally bounded degree*, that is

$$\sup_{x\in\mathbb{V}} m(x) < \infty, \quad (3)$$

where $m(x)$ is the *number of nearest neighbors*. The infinite-volume energy functional (1) cannot be defined directly as a mathematical object and is represented by the local (i.e. indexed by finite volumes $\Lambda \Subset \mathbb{V}$) Hamiltonian below.

Rigorously, μ can be given using *DLR (Dobrushin-Lanford-Ruelle) equilibrium equations*: $\forall \Lambda \Subset \mathbb{V}$, and $B \in \mathcal{B}(\Omega)$

$$\mu\pi_\Lambda(B) := \int_\Omega \pi_\Lambda(B \mid \xi)\mu(d\xi) = \mu(B),$$

for the family of *local specifications*

$$\pi_\Lambda(B|\xi) = \frac{1}{Z_\Lambda(\xi)}\int_{\Omega_\Lambda} e^{-\beta E_\Lambda(\sigma_\Lambda|\xi_{\Lambda^c})} \mathbf{1}_B(\sigma_\Lambda,\xi_{\Lambda^c}) \prod_{x\in\Lambda} d\sigma(x)$$

and the *local Hamiltonian* corresponding to $\Lambda \Subset \mathbb{V}$ and boundary condition ξ

$$\begin{aligned} E_\Lambda(\sigma_\Lambda \mid \xi_{\Lambda^c}) := & \sum_{\substack{x,y\in\mathbb{V}\\ x\sim y}} W_{xy}(\sigma(x),\sigma(y)) \\ & + \sum_{x\in\Lambda, y\in\Lambda^c} W_{xy}(\sigma(x),\xi(y)) \\ & + \sum_{x\in\Lambda} U_x(\sigma(x)) \end{aligned}$$

and, for $\xi_{\Lambda^c} := (\xi(y))_{y\in\Lambda^c}$,

$$Z_\Lambda(\xi) := \int_{\Omega_\Lambda} \exp\{-\beta E_\Lambda(\sigma_\Lambda|\xi_{\Lambda^c})\} \prod_{x\in\Lambda} d\sigma(x)$$

is the normalizing constant. As is typical for systems with *unbounded spins*, we have to restrict ourselves to certain classes of reasonable configurations $\sigma \in \Omega$. Let us fix any initial point $x_0 \in \mathbb{V}$ and define the family of Banach spaces

$$\Omega_\gamma := \left\{ \sigma \in \Omega \,\middle|\, \|\sigma\|_\gamma^R := \sum_{x\in\mathbb{V}} e^{-\gamma d(x,x_0)} |\sigma(x)|^R < \infty \right\},$$

indexed by $\gamma > \gamma_0$ with $\gamma_0 \geq \log m$. Then the set of (exponentially) tempered configurations is defined as

$$\Omega^t := \bigcap_{\gamma>\gamma_0} \Omega_\gamma.$$

And we define the set of tempered Gibbs measures as Gibbs measures supported by Ω^t.

This study shows that the set of tempered Gibbs measures is not empty. Additionally, we study a non-trivial example for the existence of tempered Gibbs measures. The crucial new issue is that we consider Hamiltonians with possibly *unbounded* interaction strength. We also give an existence result for *ferromagnetic* systems on *general* graphs. A new and very important issue is that we consider any graph $G(\mathbb{V},\mathbb{E})$ with possibly *unbounded degree*, i.e. $\sup_{x\in\mathbb{V}} m(x) \leq +\infty$.

4. Uniqueness of Gibbs measures

With additional assumptions we have the following uniqueness theorem.

Theorem 4.1. *Suppose that the parameters of the considered system satisfy the inequality*

$$\frac{b_+^2 e^{2\beta\delta}}{b_-^2 + \frac{a^2}{Jm}} < 1. \quad (4)$$

Then

$$|\mathcal{G}^t| = 1.$$

The condition (4) immediately fulfills the Dobrushin's uniqueness criterion.

In the case of *unbounded* interaction strength we consider a lattice and graph model which provides points with *very strong interaction strength* with their neighbors. In the next theorems we give uniqueness results with sufficient conditions for the lattice and graph case which uses a generalized version of Dobrushin's uniqueness criterion. The main and unexpected point here is that the sufficient conditions are independent of the unbounded interaction strength.

Theorem 4.2. *Suppose that the parameters of the lattice $\mathbb{Z}^d$ satisfy the inequality*

$$\frac{2d^2 J e^{2(2d+1)\beta\delta}}{a^2} < 1. \quad (5)$$

Then independently of the additional interaction intensities I_i the set $\mathcal{G}^t$ is a singleton, i.e.,

$$|\mathcal{G}^t| = 1.$$

Theorem 4.3. *Suppose that the parameters of the graph $G(\mathbb{V},\mathbb{E})$ satisfy the inequality*

$$\frac{m^2 J e^{2(m+1)\beta\delta}}{a^2} < 1. \quad (6)$$

Then independently of the additional interaction intensities I_i the set $\mathcal{G}^t$ is a singleton, i.e.,

$$|\mathcal{G}^t| = 1.$$

We also give a *general uniqueness criterion* for *ferromagnetic scalar models*. In contrast to the standard moment problems, such criterion uses information about the first moments of the Gibbs measures only.

5. Phase transition of Gibbs measures

We present a new method showing phase transition in *unbounded* spin systems by using the so-called *Wells inequality*. Our aim is to develop a method of proving phase transitions for systems of *unbounded continuous spins on infinite graphs*. In this area we have the following important new Theorem, which gives the critical inverse temperature of a ferromagnetic model with the *spin space* $\mathbb{R}$ and *double-well potentials* on *general trees*.

Theorem 5.1. *Let T be a general tree with the branching number $\mathrm{br}(T) < \infty$. Let us consider the scalar ferromagnetic model*

$$E(\sigma) = -\sum_{\substack{x,y\in\mathbb{V}\\ x\sim y}} J_{xy}\sigma(x)\sigma(y) + \sum_{x\in\mathbb{V}} U(\sigma(x)), \quad (7)$$

where, for $x \neq y$, $J_{xy} \geq 0$ is the interaction strength between each x and y and $U(\sigma(x))$ an even self interaction potential *(double-well potential) defined as*

$$U(\sigma(x)) := \sigma(x)^4 - \kappa\sigma(x)^2, \quad \sigma(x) \in \mathbb{R},$$

for a given $\kappa > 0$. Then the critical inverse temperature of this system equals

$$\beta_c = \frac{8\coth^{-1} br(T)}{\kappa J}.$$

This means that we have phase transition for $\beta > \beta_c$, which is the case for big κ or big J.

References

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