G-Convergence of Monotone Operators in Varying Spaces

Bielefeld University
Stochastics and Real World Models
International Graduate College

Jonas M. Tölle

International Graduate College "Stochastics and Real World Models", Beijing — Bielefeld

jtoelle1@math.uni-bielefeld.de

1. Motivation

Classically, for a sequence of convex variational problems, it is natural to link the convergence of minimizers (or solutions) to some appropriate kind of variational convergence of the energy functionals. For maximal monotone operators, the notion of G-convergence is equivalent to the strong convergence of solutions (of the associated operator equations) and to strong convergence of the associated flows and resolvents. The precise equivalent to it in the realm of convex functionals is the variational convergence known as the Mosco convergence.

Many variational problems are associated naturally to some specific Banach space. These spaces could depend on some kind of "parameter", as e.g. the space $L^p(\Omega;\mu)$ depends on p, the measure μ and the domain Ω. We have been considering two questions:

- Can we give a criterion for a point $x \in L^{p_1}(\Omega_1;\mu_1)$ being near a point $y \in L^{p_2}(\Omega_2;\mu_2)$, in the situation that, say $p_1 \to p_2$ in $\mathbb{R}$, $\mu_1 \rightharpoonup \mu_2$ in the weak sense and $\Omega_1 \to \Omega_2$ e.g. in the Kuratowski sense?
- If we attach a convex energy functionals to each of these spaces, can we prove Mosco convergence, when the parameters converge suitably?

Both questions can be answered positively. The theory of varying Banach spaces has been developed successively in [4, 8, 5, 7]. In Hilbert space, results on Mosco convergence have been obtained e.g. in [2, 3]. For the first time, the nonlinear situation has been considered in [7]. We mention that since 1990 or so a special situation of varying spaces is well-known in the theory of homogenization under the name "two-scale convergence". As a matter of fact, it is embedded into our framework. For applications of G-convergence to stochastics, we mention Ioana Ciotir's work [1].

2. Varying Spaces

Let E_n, $n \in \mathbb{N}$, E be Banach spaces. Suppose there is a sequence of linear maps $\{\Phi_n : C \subset E \to E_n\}$ defined on a dense linear subspace $C \subset E$ such that

$$\lim_n \|\Phi_n(\varphi)\|_{E_n} = \|\varphi\|_E \quad \forall \varphi \in C. \tag{1}$$

We define a convergence relation by saying that $u_n \in E_n$, $n \in \mathbb{N}$ converges to $u \in E$ if and only if for any sequence $\{\varphi_m\} \subset C$ such that $\lim_m \|u - \varphi_m\|_E = 0$ it holds that

$$\lim_m \overline{\lim_n} \|u_n - \Phi_n(\varphi_m)\|_{E_n} = 0. \tag{2}$$

In fact, the convergence defined in (2) is a sequential convergence and hence generates a topology on the disjoint union of spaces

$$\bigsqcup_n E_n \sqcup E.$$

It is called the *asymptotic topology* and it has the following properties.

- E_n, $n \in \mathbb{N}$, E are closed and the relative topologies of E_n, $n \in \mathbb{N}$, E coincide with the original strong topologies. Also, E_n, $n \in \mathbb{N}$ are open.
- For any $u \in E$ there exists a sequence $\{u_n\}$ such that $u_n \in E_n$, $n \in \mathbb{N}$ and $\lim_n u_n = u$.
- For any sequence $\{u_n\}$, with $u_n \in E_n$, $n \in \mathbb{N}$ and any $u \in E$ the following statement holds:
  If $\lim_n u_n = u$, then
  $$\lim_n \|u_n\|_{E_n} = \|u\|_E\,.$$
- For any two sequences $\{u_n\}$, $\{v_n\}$ with $u_n, v_n \in E_n$, $n \in \mathbb{N}$ and any $u \in E$ the following statement holds:
  If $\lim_n u_n = u$ and $\lim_n \|u_n - v_n\|_{E_n} = 0$, then $\lim_n v_n = u$.
- For any two sequences $\{u_n\}$, $\{v_n\}$ with $u_n, v_n \in E_n$, $n \in \mathbb{N}$ and any two $u, v \in E$ and any $\alpha, \beta \in \mathbb{R}$ the following statement holds:
  If $\lim_n u_n = u$ and $\lim_n v_n = v$, then $\lim_n \left[\alpha u_n + \beta v_n\right] = \alpha u + \beta v$.

Weak and weak* Banach space topologies are also treated with the help of so-called "asymptotic duality". The complete framework is developed in [7].

3. Variational Convergence

Let $\{F_n\}$ be a sequence of proper, convex and l.s.c. functionals on some Banach space E (with dual space E^*). Let F be a proper, convex, l.s.c. functional on E. Denote by $\partial F_n : E \to E^*$, $\partial F : E \to E^*$ the associated subdifferentials, which are maximal cyclically monotone operators.

We say that $F_n \to F$ *Mosco*, if

- $\forall\{u_n\}$, $\forall u$ such that $u_n \rightharpoonup u$ weakly in E it holds that $\underline{\lim}_n F_n(u_n) \geq F(u)$,
- $\forall v$ $\exists\{v_n\}$ such that $v_n \to v$ strongly in E and $\overline{\lim}_n F_n(v_n) \leq F(v)$.

When E is reflexive, by a well-known theorem due to Umberto Mosco and Hedy Attouch, $F_n \to F$ Mosco if and only if $\partial F_n \to \partial F$ in the G-*sense*, i.e.,

- $\forall [u, f] \in \partial F$ $\exists\{u_n\}$, $\exists\{f_n\}$ such that $[u_n, f_n] \in \partial F_n$, $u_n \to u$ strongly in E, $f_n \to f$ strongly in E^*.

We show in [7] that the above Mosco–Attouch Theorem also holds in any dual pair of reflexive varying spaces.

4. Weighted p-Laplace Operators

Let $d \in \mathbb{N}$. Let $\{p_n\} \subset (1,\infty)$ with $\lim_n p_n = p \in (1,\infty)$. Set $q_n := p_n/(p_n - 1)$. Let $w_n, w \in L^1_{\text{loc}}(dx)$, be weights with full support in $\mathbb{R}^d$ such that

$$w_n^{1-q_n},\ w^{1-q} \in L^1_{\text{loc}}(dx). \tag{3}$$

Let $L^{p_n}_{w_n} := L^{p_n}(\mathbb{R}^d; w_n\,dx)$, $L^p_w := L^p(\mathbb{R}^d; w\,dx)$ and let $H^{1,p_n}_{w_n,0}$, $H^{1,p}_{w,0}$ be the first order weighted strong p_n resp. p-Sobolev spaces, i.e. the closures of C_0^∞ w.r.t. the weighted Sobolev norm.

Define the energies

$$F_n(u) := \frac{1}{p_n}\int |\nabla u|^{p_n} w_n\,dx,\quad F(u) := \frac{1}{p}\int |\nabla u|^p w\,dx.$$

Set $F_n(u) := +\infty$ if $u \in L^{p_n}_{w_n} \setminus H^{1,p_n}_{w_n,0}$ and $F(u) := +\infty$ if $u \in L^p_w \setminus H^{1,p}_{w,0}$.

Suppose that

$$w_n \rightharpoonup w \quad \text{weakly in } L^1_{\text{loc}}(dx). \tag{4}$$

Then the sequence of maps $\{\Phi_n : C_0^\infty \subset L^p_w \to L^{p_n}_{w_n}\}$, where $\Phi_n(\varphi) := \varphi$, $\varphi \in C_0^\infty$, defines an asymptotic topology on $\bigsqcup_n L^{p_n}_{w_n} \sqcup L^p_w$.

Theorem 1. *Suppose that $H^{1,p}_{w,0} = W^{1,p}_w$ (the weak weighted Sobolev space) and that*

$$\sup_n \left[\int_B {w_n}^{1-q_n}\,dx\right]^{(p_n-1)/p_n} < \infty \quad \textit{for all balls } B \Subset \mathbb{R}^d, \tag{5}$$

$$w_n^{1-q_n} \rightharpoonup w^{1-q} \quad \textit{weakly in } L^1_{\text{loc}}(dx), \tag{6}$$

Then $F_n \to F$ Mosco in $\bigsqcup_n L^{p_n}_{w_n} \sqcup L^p_w$.

As a result,

$$\operatorname{div}\left[w_n|\nabla u|^{p_n-2}\nabla u\right] \to \operatorname{div}\left[w|\nabla u|^{p-2}\nabla u\right],$$

in the G-sense. The p-Laplace operator has numerous applications in physics, e.g. nonlinear diffusions, non-Newtonian fluids, flows in porous media and plasma physics.

5. Generalized Porous Medium Operators

We follow the framework of [6]. Let $(E,\mathscr{B})$ be a measurable space. Let $\Phi \in C^1(\mathbb{R})$ be a nice Young function satisfying Δ_2 and ∇_2 growth conditions (e.g. $\Phi(x) = \frac{1}{p}|x|^p$, $1 < p < \infty$). Let L_n, L be transient linear Dirichlet operators in $L^2(E;\mu_n)$, $L^2(E;\mu)$, where μ_n, μ are their symmetrizing (probability) measures. Suppose that they have full support on E. Denote by $(\mathscr{E}_n,\mathscr{F}_n)$, $(\mathscr{E},\mathscr{F})$ the Dirichlet spaces of L_n, L. Denote by $\mathscr{F}_n^*$, $\mathscr{F}^*$ the abstract Green spaces of L_n, L. Let $V_n := \mathscr{F}_n^* \cap L^\Phi(E;\mu_n)$, $V := \mathscr{F}^* \cap L^\Phi(E;\mu)$, where $L^\Phi(E;\mu_n)$, $L^\Phi(E;\mu)$ denote the Orlicz spaces.

Define the convex functionals

$$G_n(u) := \int_E \Phi(|u|)\,d\mu_n,\quad G(u) := \int_E \Phi(|u|)\,d\mu.$$

Set $G_n(u) := +\infty$ if $u \in \mathscr{F}_n^* \setminus V_n$ and $G(u) := +\infty$ if $u \in \mathscr{F}^* \setminus V$.

Theorem 2. *Suppose that $\mu_n \rightharpoonup \mu$ in the weak sense. Suppose that L has a core $C \subset C_b(E)$. Suppose that $\bigsqcup_n \mathscr{F}_n^* \sqcup \mathscr{F}^*$ has an asymptotic topology. Suppose that $L(C)$ is dense in V. Then $G_n \to G$ Mosco in $\bigsqcup_n \mathscr{F}_n^* \sqcup \mathscr{F}^*$.*

Concrete examples for the conditions to hold are given in [7], in particular, it is sufficient that $L(C) \subset C_b(E)$ densely and $\mathscr{E}_n(\varphi) \to \mathscr{E}(\varphi)$ for all $\varphi \in C$. As a result,

$$L_n\left[\Phi'(|u|)\operatorname{sign}(u)\right] \to L\left[\Phi'(|u|)\operatorname{sign}(u)\right],$$

in the G-sense. We remark that a possible limit is the classical porous medium operator $\Delta\left[|u|^{p-2}u\right]$, $p > 2$, on $\mathbb{R}^d$ with $d \geq 3$. A number of physical applications for the porous medium operator is known, such as to describe processes involving a flow of fluid through a porous medium. As other applications, we mention nonlinear diffusion and heat transfer, plasma physics, lubrication and material science.

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