Introduction to Gradient Flows in Metric Spaces (II)

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Preface

The aim of these Lectures Notes is to provide a gentle introduction to the theory of gradient flows in metric spaces developed in the first part of the book of Ambrosio-Gigli-Savaré [AGS]. In contrast to [LN1] we do not use the notion of (local) slope of a functional defined on a metric space. As in [LN1] we use "Crandall-Liggett" type estimates. This approach is somewhat simpler than the one used in [AGS] but it does not give the optimal rate of convergence of the approximation scheme obtained in [AGS].

The main result of these notes, Theorem 4.1 is a special case ($\alpha = 0$; in [AGS]: $\lambda = 0$) of a result of [CD2]. Also in these notes we use recent results of [CD1] which simplify the approach of [LN1]. For the sake of completeness we give the corresponding estimates of Crandall-Liggett [CL71] when the operator A is m-accretive in a Banach space $(X, \|\cdot\|)$.

The main difference between the two cases, $(X, \|\cdot\|)$ Banach space and the situation of Theorem 4.1 is that the "resolvent" operators J_h are contractive in the accretive case but not (in general) under the assumptions of Theorem 4.1. Finally it should be observed that if X is a real Hilbert space and the function $\phi\colon X \to (-\infty, \infty]$ is proper, lower semicontinuous and convex, then, the subdifferential $\partial\phi$ is m-accretive, hence the first approach can be used, and the function ϕ satisfies the assumption (A) of Theorem 4.1, providing another approach to the same problem. Combining Lemma 3.2, Proposition 3.1 and Theorem 4.1, one obtains the existence and uniqueness part of the proof of Theorem 4.0.4 of [AGS] as well as other properties of solutions to the evolution variational inequality (4.0.13) of [AGS].

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1 An "Evolution Variational Inequality" on a metric space

The aim of this section is to introduce an evolution variational inequality (EVI) on a metric space which will be the main subject of these notes. We motivate the definition by means of two examples. We consider in Section 1.1 a gradient flow in $\mathbb{R}^N$ with the euclidean metric and in Section 1.2 a "heat flow" in $L^2(\mathbb{R}^n)$. We show in both cases that the corresponding solutions satisfy (EVI). In this reformulation of the problem, the linear structure is not used. In Section 1.3 the definition of a solution to (EVI) is given and in Section 1.4 we prove an a priori estimate which among other things guarantees the uniqueness for the corresponding initial value problem.

1.1 Gradient flows in $\mathbb{R}^N$

Let $H := \mathbb{R}^N$ with the euclidean metric and let $\phi : \mathbb{R}^N \to \mathbb{R}$ be *differentiable*. We consider the differential equation

$$\text{(DE)} \qquad \dot{u}(t) = -\nabla\phi(u(t)), \quad t \in J,$$

where J is an open interval of $\mathbb{R}$, $\dot{u}(t) := \frac{d}{dt}u(t)$ and

$$\nabla\phi(x) := \begin{bmatrix} \frac{\partial\phi}{\partial x_1}(x) \\ \vdots \\ \frac{\partial\phi}{\partial x_N}(x) \end{bmatrix}, \quad x \in \mathbb{R}^N.$$

A solution of (DE) is a function $u : J \to \mathbb{R}^N$ which is differentiable and satisfies (DE). Moreover

$$\text{(1.1)} \qquad \frac{d}{dt}\phi \circ u(t) = \langle \nabla\phi(u(t)), \dot{u}(t)\rangle = -\big|\nabla\phi(u(t))\big|^2 = -\big|\dot{u}(t)\big|^2 \le 0, \quad t \in J,$$

hence $\phi \circ u$ is *nonincreasing* on J.

In this section we require in addition that $\nabla\phi$ is Lipschitz continuous. We recall

Definition 1.1. Let (X_i, d_i), $i = 1, 2$, be two metric spaces and let $F : X_1 \to X_2$. The map F is called *Lipschitz continuous* if there exists $M \ge 0$ such that

$$\text{(1.2)} \qquad d_2(F(x), F(y)) \le M d_1(x, y)$$

for every $x, y \in X_1$.

If $X_1 = X_2$, the map F is called a *contraction* (in X_1) when $M = 1$ and a *strict contraction* (in X_1) when $M < 1$. The smallest constant for which (1.2) holds will be denoted by $[F]_{\text{Lip}}$.

Remark 1.1. For any Lipschitz continuous map F we have

$$\text{(1.3)} \qquad [F]_{\text{Lip}} = \sup\left\{ \frac{d_2(F(x), F(y))}{d_1(x, y)} : x, y \in X_1,\ x \ne y \right\}.$$

For any map $F : X_1 \to X_2$ we can take the RHS (right-hand side) of (1.3) as the definition of $[F]_{\mathrm{Lip}}$. So F is Lipschitz continuous (notation: $F \in \mathrm{Lip}(X_1, X_2)$) iff $[F]_{\mathrm{Lip}} < \infty$.

Returning to (DE), with $[\nabla\phi]_{\mathrm{Lip}} < \infty$, we recall the following global existence and uniqueness theorem. For every $x_0 \in \mathbb{R}^N$, there exists a unique solution u of (DE) on $J = \mathbb{R}$ such that $u(0) = x_0$.

We shall show that any solution of (DE) satisfies an evolution variational inequality on $X = \mathbb{R}^N$ involving only the metric of $\mathbb{R}^N$ and not its linear structure.

We proceed in 5 steps.

Step 1. "$\nabla\phi$ is quasi-monotone".

Definition 1.2. Let $(H, \langle,\rangle)$ be a real Hilbert space and $D(A)$ be a nonempty subset of H. A map (operator) $A : D(A) \subset H \to H$ is called *monotone* if

$$(1.4)\qquad \langle Ax - Ay, x - y\rangle \geq 0 \quad \text{for all} \quad x, y \in D(A),$$

and *quasi-monotone* if there exists $\alpha \in \mathbb{R}$ such that $A - \alpha I$ is monotone, equivalently

$$(1.5)\qquad \langle Ax - Ay, x - y\rangle \geq \alpha|x - y|^2 \quad \text{for all} \quad x, y \in D(A).$$

If α in (1.5) can be chosen positive then A is called *strongly monotone.*

Setting

$$(1.6)\qquad \alpha(F) := \inf\left\{ \frac{\langle Ax - Ay, x - y\rangle}{|x - y|^2} : x, y \in D(A),\ x \neq y \right\}$$

(possibly $-\infty$) we see that A is quasi-monotone iff $\alpha(F) > -\infty$, and A is monotone iff $\alpha(F) \geq 0$.

Finally we observe that if $F : H \to H$ is Lipschitz continuous, then

$$\langle F(x) - F(y), x - y\rangle \geq -|F(x) - F(y)|\,|x - y| \geq -[F]_{\mathrm{Lip}}|x - y|^2, \quad x, y \in H;$$

hence

$$(1.7)\qquad \alpha(F) \geq -[F]_{\mathrm{Lip}}$$

and F is quasi-monotone, since $F - \beta I$ is monotone for every $\beta \leq \alpha(F)$.

Step 2. "ϕ is quasi-convex".

In view of Step 1 there exists $\alpha \in \mathbb{R}$ such that $\nabla\phi - \alpha I$ is monotone. Setting

$$(1.8)\qquad e(x) := \tfrac{1}{2}|x|^2 \quad \text{for} \quad x \in \mathbb{R}^N$$

and noticing that $\nabla e(x) = x$, $x \in \mathbb{R}^N$, i.e. $\nabla e = I$, the identity in $\mathbb{R}^N$, we see that $\nabla(\phi - \alpha e) = \nabla\phi - \alpha\nabla e = \nabla\phi - \alpha I$ is monotone.

Set

$$(1.9)\qquad \psi := \phi - \alpha e,$$

then $\psi : \mathbb{R}^N \to \mathbb{R}$ is differentiable and $\nabla\psi$ is monotone, hence ψ is *convex* as a consequence of the following lemma.

Lemma 1.1. *Let $\psi : \mathbb{R}^N \to \mathbb{R}$ be everywhere differentiable with $\nabla\psi$ monotone. Then ψ is convex.*

Proof. Let $\nabla\psi$ be monotone and let $x, y \in \mathbb{R}^N$, $t \in \mathbb{R}$. Set

$$\alpha(t) := \psi((1-t)x + ty) - (1-t)\psi(x) - t\psi(y).$$

Then $\alpha(0) = \alpha(1) = 0$ and α is differentiable,

$$\alpha'(t) = \langle \nabla\psi((1-t)x + ty), y - x\rangle + \psi(x) - \psi(y).$$

Let $t_1 < t_2$. Observe that $[(1-t_2)x + t_2 y] - [(1-t_1)x + t_1 y] = (t_2 - t_1)(y - x)$. We have

$$\begin{aligned}\alpha'(t_2) - \alpha'(t_1) = \langle \nabla\psi((1-t_2)x + t_2 y) - \nabla\psi((1-t_1)x + t_1 y),& \\ [(1-t_2)x + t_2 y] - [(1-t_1)x + t_1 y]\rangle \cdot \frac{1}{t_2 - t_1} &\geq 0.\end{aligned}$$

Hence α' is nondecreasing. If α had a positive maximum $\xi \in (0,1)$, then $\alpha'(\xi) = 0$ and in view of the mean value theorem α would be nonincreasing for $t < \xi$ and nondecreasing for $t > \xi$. A contradiction. Hence $\alpha(t) \leq 0$ for $t \in [0,1]$ and ψ is convex. □

We can rewrite (DE) as

$$-\dot{u}(t) - \alpha u(t) = \nabla\psi(u(t)), \quad t \in J, \tag{1.10}$$

where ψ is convex.

Step 3. "$\nabla\psi(x)$ is a subgradient of ψ at x".

Let $x, h \in \mathbb{R}^N$. Since ψ is differentiable at x, we have

$$\psi(x+h) - \psi(x) = \langle \nabla\psi(x), h\rangle + o(|h|),$$

hence

$$\lim_{t\downarrow 0} \frac{1}{t}\big[\psi(x+th) - \psi(x)\big] = \langle \nabla\psi(x), h\rangle\,.$$

Since ψ is convex, the function $t \mapsto \psi(x+th)$ is also convex, hence the function

$$0 < t \mapsto \frac{1}{t}\big(\psi(x+th) - \psi(x)\big)$$

is nondecreasing. Therefore

$$\langle \nabla\psi(x), h\rangle = \lim_{t\downarrow 0} \frac{1}{t}\big(\psi(x+th) - \psi(x)\big) \overset{(t=1)}{\leq} \psi(x+h) - \psi(x).$$

Equivalently, we have

$$\psi(z) \geq \psi(x) + \langle y, z - x\rangle \quad \text{for all} \quad z \in \mathbb{R}^N, \tag{1.11}$$

where $y = \nabla\phi(x)$. Any $y \in \mathbb{R}^N$ satisfying (1.11) is called a *subgradient* of ψ at x.

Actually $y = \nabla\psi(x)$ is the only subgradient of ψ at x. Indeed, from (1.11), choosing $z = x + th$, $t > 0$, $h \in \mathbb{R}^N$, we deduce

$$\frac{1}{t}(\psi(x+th) - \psi(x)) \geq \langle y, h\rangle\,.$$

By taking the limit as $t \to 0$ we get $\langle \nabla\psi(x), h\rangle \geq \langle y, h\rangle$. Replacing h by $-h$ we obtain equality. Choosing $h = y - \nabla\psi(x)$ we arrive at $y = \nabla\psi(x)$.

Step 4. "Some computation".

Combining (1.10) and (1.11) we obtain

$$\langle -\dot{u}(t) - \alpha u(t), z - u(t)\rangle + \psi(u(t)) \le \psi(z)$$

for every $x, z \in \mathbb{R}^N$, $t \in J$.

Setting

$$d^2(x,y) := (d(x,y))^2 = |x-y|^2, \quad x, y \in \mathbb{R}^N, \tag{1.12}$$

and using the definition of ψ, we obtain

$$\frac{1}{2}\frac{d}{dt}d^2(u(t),z) + \alpha|u(t)|^2 - \alpha\,\langle u(t), z\rangle + \phi(u(t)) - \frac{\alpha}{2}|u(t)|^2 \le \phi(z) - \frac{\alpha}{2}|z|^2.$$

Hence for all $t \in J$, all $z \in \mathbb{R}^N$, we have

$$\frac{1}{2}\frac{d}{dt}d^2(u(t),z) + \frac{\alpha}{2}\,d^2(u(t),z) + \phi(u(t)) \le \phi(z). \tag{1.13}$$

Step 5. "Integration by parts".

This step consists of replacing the pointwise derivative in (1.13) by a "weak derivative". Multiplying (1.13) by a *nonnegative* test function $\eta \in C_c^\infty(J)$ ($C^\infty(J)$ with compact support in J), and integrating by parts we arrive at

$$-\int_J \frac{1}{2}\,d^2(u(t),z)\dot{\eta}(t)\,dt + \frac{\alpha}{2}\int_J d^2(u(t),z)\eta(t)\,dt \le \int_J \big[\phi(z) - \phi\circ u(t)\big]\eta(t)\,dt, \tag{EVI}$$

for every nonnegative $\eta \in C_c^\infty(J)$.

Observe that the integrals in (EVI) are well defined since $u \in C(J;\mathbb{R}^N)$, $\phi\circ u \in C(J;\mathbb{R})$ as well as $t \mapsto d^2(u(t),z)$, and the supports of η, $\dot{\eta}$ are compact in J.

Notice that (EVI) makes sense if we replace $(\mathbb{R}^N, d)$ by an arbitrary metric space (X, d), using the notation (1.12), and requiring $\phi \in C(X;\mathbb{R})$ and the solution u to be *continuous* on J with values in X. In Section 1.2 we shall consider an example where we need to weaken the condition on the functional ϕ which requires a stronger assumption for the definition of a solution.

Problem 1.1. Let $\phi : \mathbb{R}^n \to \mathbb{R}$ be everywhere differentiable. Show that if $\nabla\phi(x) = Ax$ for all $x \in \mathbb{R}^N$ for some $N \times N$ matrix A, then the matrix A is symmetric. Show that in this case $\phi(x) = \frac{1}{2}\langle Ax, x\rangle + \text{constant}$. Find a characterization of $[\nabla\phi]_{\text{Lip}}$ (1.3) and $\alpha(\nabla\phi)$ (1.6) in terms of the eigenvalues of A.

Problem 1.2. Let $(H, \langle,\rangle)$ be a real Hilbert space and let $A \in \mathcal{L}(H)$ be a bounded linear operator on H. We recall that $\sum_{k=0}^{n} \frac{1}{k!}A^k$ converges in $\mathcal{L}(H)$ (with respect to the operator norm) as $n \to \infty$. We denote the limit by e^A, the exponential of A. Moreover, given x_0 and $f \in H$, the function $u : \mathbb{R} \to H$ defined by

$$u(t) = e^{tA}x_0 + \int_0^t e^{sA}f\,ds, \quad t \in \mathbb{R}, \tag{1.14}$$

is continuously differentiable ($C^1(\mathbb{R};H)$)), i.e., $\lim \frac{1}{h}(u(t+h) - u(t))$ exists in H for all $t \in \mathbb{R}$ and is denoted by $\dot{u}(t)$, and $\mathbb{R} \ni t \mapsto \dot{u}(t) \in C(\mathbb{R};H)$. Moreover, u satisfies the linear differential equation in H:

$$\dot{u}(t) + Au(t) = f, \quad t \in \mathbb{R}, \tag{LDE}$$

together with $u(0) = x_0$.

Show that if A is symmetric, i.e.,

$$\langle Ax, y\rangle = \langle x, Ay\rangle \quad \text{for all} \quad x, y \in H,$$

then the function u defined in (1.14) satisfies (EVI) with $J := \mathbb{R}$,

$$\phi(x) := \tfrac{1}{2}\langle Ax, x\rangle - \langle f, x\rangle, \quad x \in H,$$

and

$$\alpha := \inf_{\substack{x\in H\\ x\neq 0}} \frac{\langle Ax, x\rangle}{|x|^2} = \alpha(A).$$

1.2 Heat flow in $L^2(\mathbb{R}^N)$ as a gradient flow in $L^2(\mathbb{R}^N)$

The aim of this section is to show that the heat flow in $L^2(\mathbb{R}^n)$ can be viewed as a "gradient flow" in $L^2(\mathbb{R}^n)$ with respect to the "Dirichlet" functional.

Let

$$p(t, x) := \frac{1}{(2\pi t)^{N/2}}\, e^{-|x|^2/(2t)}, \quad t > 0,\ x \in \mathbb{R}^N, \tag{1.15}$$

be the density of the gaussian probability measure on $\mathbb{R}^N$ with mean 0 and second moment Nt, i.e.,

$$\int_{\mathbb{R}^N} p(t, x)\, dx = 1, \qquad t > 0,$$
$$\int_{\mathbb{R}^N} x_k p(t, x)\, dx = 0, \qquad k = 1, \ldots, N,$$
$$\int_{\mathbb{R}^N} |x|^2 p(t, x)\, dx = Nt, \qquad t > 0.$$

Let $f_0 : \mathbb{R}^N \to \mathbb{R}$ be a (Lebesgue) measurable function satisfying $\int_{\mathbb{R}^N} |f_0|^2\, dx < \infty$. Let $t > 0$ and $x \in \mathbb{R}^N$. Then

$$\int_{\mathbb{R}^N} p(2t, x - y)|f_0(y)|\, dy < \infty,$$

hence

$$v(t, x) := \int_{\mathbb{R}^N} p(2t, x - y) f_0(y)\, dy \tag{1.16}$$

is well-defined. We recall without proofs several facts about the function

$$v : (0, \infty) \times \mathbb{R}^n \to \mathbb{R}$$

which will be used later.

The function v is infinitely differentiable in $(0, \infty) \times \mathbb{R}^N$ (notation: $u \in C^\infty((0, \infty) \times \mathbb{R}^N)$). Clearly, if f_0 is replaced by g_0 such that $g_0 = f_0$ a.e. in $\mathbb{R}^N$ then the RHS of (1.16) defines the same function v so (1.16) defines a map from $L^2(\mathbb{R}^N)$ into $C^\infty((0, \infty) \times \mathbb{R}^N)$. The function v satisfies the "heat equation"

$$\frac{\partial}{\partial t} v(t, x) = \Delta v(t, x), \quad (t, x) \in (0, \infty) \times \mathbb{R}^N, \tag{1.17}$$

where $\Delta v = \sum_{k=1}^{N} \frac{\partial^2}{\partial x_k^2} v$.

Set

$$u(t)(x) := v(t,x), \quad t > 0, \; x \in \mathbb{R}^N. \tag{1.18}$$

For t fixed $u(t) \in C^\infty(\mathbb{R}^N)$ and $\int_{\mathbb{R}^N} \big(u(t)(x)\big)^2 \, dx < \infty$, hence $u(t)$ can be viewed as an element of $L^2(\mathbb{R}^N)$. Moreover, the map defined by $(0,\infty) \ni t \mapsto u(t) \in L^2(\mathbb{R}^N)$ will be denoted by u. Then it appears that $u \in C((0,\infty); L^2(\mathbb{R}^N))$.

Even more, for every $t > 0$ we have

$$\lim_{h \to 0} \int_{\mathbb{R}^N} \Big| \frac{1}{h}\big(u(t+h)(x) - u(t)(x)\big) - \frac{\partial}{\partial t} v(t,x) \Big|^2 \, dx = 0. \tag{1.19}$$

This shows that the map $u : (0,\infty) \to L^2(\mathbb{R}^N)$ is (strongly) differentiable in $L^2(\mathbb{R}^N)$. We denote its derivative at t by $\dot{u}(t)$. Similarly $\dot{u}$ is differentiable on $(0,\infty)$ and $\ddot{u}(t)(x) = \frac{\partial^2}{\partial t^2} v(t,x)$, $(t,x) \in (0,\infty) \times \mathbb{R}^N$. In fact u is infinitely differentiable (notation: $u \in C^\infty((0,\infty); L^2(\mathbb{R}^N))$. Moreover, for every $t > 0$ the function $u(t) \in L^2(\mathbb{R}^N)$ possesses weak derivatives of all order which belong to $L^2(\mathbb{R}^N)$, i.e., $u(t) \in W^{k,2}(\mathbb{R}^N)$ for all $k \in \mathbb{N}$. Thus for every multi-index $(\alpha_1, \ldots, \alpha_N)$ $x \mapsto \big(\frac{\partial^{\alpha_1}}{\partial x^{\alpha_1}} \cdots \frac{\partial^{\alpha_N}}{\partial x^{\alpha_N}} v(t,x)\big)$ is an element of the equivalence class of

$$\frac{\partial^{\alpha_1}}{\partial x^{\alpha_1}} \cdots \frac{\partial^{\alpha_N}}{\partial x^{\alpha_N}} u(t)$$

for every $t > 0$.

As a consequence of (1.17) one shows that

$$\dot{u}(t) + Au(t) = 0, \quad t > 0, \tag{1.20}$$

where $A : W^{2,2}(\mathbb{R}^n) \subset L^2(\mathbb{R}^N) \to L^2(\mathbb{R}^N)$ is defined by

$$Af := -\sum_{k=1}^{N} \frac{\partial^2}{\partial x_k^2} f,$$

where $\frac{\partial}{\partial x_k}$ are weak derivatives.

Moreover, we define for $f \in W^{1,2}(\mathbb{R}^N)$

$$\nabla f := \begin{bmatrix} \frac{\partial}{\partial x_1} f \\ \vdots \\ \frac{\partial}{\partial x_N} f \end{bmatrix},$$

the "gradient" of f, where $\frac{\partial}{\partial x_k}$ are weak derivatives.

Clearly, ∇ maps $W^{1,2}(\mathbb{R}^N)$ into $\big(L^2(\mathbb{R}^N)\big)^N$. The following relation between A and ∇ is essential. We have

$$\int_{\mathbb{R}^N} (Af)(x) g(x) \, dx = \int_{\mathbb{R}^N} \sum_{k=1}^{N} \frac{\partial}{\partial x_k} f(x) \frac{\partial}{\partial x_k} g(x) \, dx \tag{1.21}$$

for every $f \in W^{2,2}(\mathbb{R}^N)$, $g \in W^{1,2}(\mathbb{R}^N)$.

Denoting by $\langle\,,\,\rangle$ the innerproduct in $L^2(\mathbb{R}^N)$, we obtain by taking the innerproduct of (1.20) with $z - u(t)$ where $z \in W^{1,2}(\mathbb{R}^N)$:

$$\begin{aligned}\langle -\dot{u}(t), z - u(t)\rangle &= \langle Au(t), z - u(t)\rangle \\ &= \langle \nabla u(t), \nabla z - \nabla u(t)\rangle = -|\nabla u(t)|^2 + \langle \nabla u(t), \nabla z\rangle \\ &\le -|\nabla u(t)|^2 + \tfrac{1}{2}|\nabla u(t)|^2 + \tfrac{1}{2}|\nabla z|^2 = -\tfrac{1}{2}|\nabla u(t)|^2 + \tfrac{1}{2}|\nabla z|^2,\end{aligned}$$

where $|\cdot|$ denotes the norm in $L^2(\mathbb{R}^N)$.

Moreover, since $u \in C^1\big((0,\infty); L^2(\mathbb{R}^N)\big)$ we have

$$\langle \dot{u}(t), z - u(t)\rangle = \frac{1}{2}\frac{d}{dt}\langle u(t) - z, u(t) - z\rangle = \frac{1}{2}\frac{d}{dt}|u(t) - z|^2 = \frac{1}{2}\frac{d}{dt}d^2(u(t), z)$$

where $d(f,g) := |f - g|$ in $L^2(\mathbb{R}^N)$.

We arrive at

$$\frac{1}{2}\frac{d}{dt}d^2(u(t), z) \le \phi(z) - \phi \circ u(t), \quad t > 0 \text{ and } z \in W^{1,2}(\mathbb{R}^N), \tag{1.22}$$

where

$$\phi(f) := \frac{1}{2}\int_{\mathbb{R}^N} |\nabla f|^2\,dx \quad \text{for} \quad f \in W^{1,2}(\mathbb{R}^N). \tag{1.23}$$

Observe that $t \mapsto d^2(u(t), z)$ is continuously differentiable as well as $t \mapsto \langle Au(t), u(t)\rangle = \langle -\dot{u}(t), u(t)\rangle$ on $(0,\infty)$.

Hence multiplying (1.22) by nonnegative test functions $\eta \in C_c^\infty(0,\infty)$ we arrive at (EVI) defined in Section 1.1 with $X = L^2(\mathbb{R}^N)$ endowed with the metric induced by the innerproduct, $\phi : W^{1,2}(\mathbb{R}^N) \subset X \to \mathbb{R}$ defined by (1.23) and $\alpha = 0$. Observe that as in Section 1.1 $t \mapsto \phi \circ u(t)$ is (at least) continuous hence locally integrable on $J := (0,\infty)$.

The function ϕ is usually called a *Dirichlet* form [K]. It is customary to define ϕ on the whole of X by setting

$$\phi(f) := \begin{cases} \frac{1}{2}\int_{\mathbb{R}^N} |\nabla f|^2\,dx & \text{for } f \in W^{1,2}(\mathbb{R}^N), \\ +\infty & \text{otherwise.}\end{cases} \tag{1.24}$$

$D(\phi) := \{f \in L^2(\mathbb{R}^N) : \phi(f) < \infty\}$ is called the *effective domain* of ϕ.

From now on we shall adopt this way of writing.

1.3 Definition of a "gradient flow" on a metric space

Motivated by the two previous examples, we shall define a *gradient flow on a metric space* as follows:

Definition 1.3. Let

- (X, d) be a metric space (not necessarily complete),
- $\phi : X \to (-\infty, +\infty]$ be proper (i.e. $D(\phi) := \{x \in X : \phi(x) < \infty\} \neq \emptyset$),
- $\alpha \in \mathbb{R}$.

Given an open interval J in $\mathbb{R}$, we say that a function $u : J \to X$ is a *gradient flow* (or a solution to (EVI)) on (X, d) (with respect to the pair (ϕ, α)), if

i) $u \in C(J; X)$,

ii) $\phi \circ u \in L^1_{\text{loc}}(J)$, i.e., $\phi \circ u|_{(a,b)} \in L^1(a, b)$ for every $a, b \in J$ such that $a < b$,

iii) u and $\phi \circ u$ satisfy (EVI) on J.

If, moreover, $J = (a, b)$ or (a, ∞) with $a, b \in \mathbb{R}$, $a < b$, and $\lim_{t\to a} u(t) = x$ exists in (X, d), then we say that the gradient flow *starts at* x or has x as *initial value*.

Remark 1.2. Clearly the function u defined in Section 1.1 is a gradient flow in $\mathbb{R}^N$ (with euclidean metric) where $J = \mathbb{R}$, $\phi \in C^1(\mathbb{R}^N; \mathbb{R})$ and $\alpha := \alpha(\nabla\phi)$. Similarly the function u defined in Section 1.2 is a gradient flow in $L^2(\mathbb{R}^N)$ where $J = (0, \infty)$, ϕ is the Dirichlet form (1.24) and $\alpha = 0$. Moreover, one can show that $\lim_{t\to 0} u(t) = f_0$ in $L^2(\mathbb{R}^N)$. Hence this gradient flow has f_0 as initial value.

Since $u(t) \in W^{2,2}(\mathbb{R}^N)$, we could have chosen for the functional ϕ, its restriction to $W^{2,2}(\mathbb{R}^N)$, namely

$$\phi_1(f) := \begin{cases} \frac{1}{2}\int_{\mathbb{R}^N}(-\Delta f) f \, dx & \text{for } f \in W^{2,2}(\mathbb{R}^N), \\ +\infty & \text{otherwise.} \end{cases} \tag{1.25}$$

It appears that ϕ is lower semicontinuous on $L^2(\mathbb{R}^N)$ but not ϕ_1. This property will play an important role in the proof of *existence* of solutions to (EVI).

1.4 An a priori estimate

The aim of this subsection is to establish an a priori estimate for solutions to (EVI). This estimate implies *uniqueness* for the corresponding initial value problem.

Proposition 1.1. *Suppose u and v are two solutions to* (EVI) (*gradient flows*) *with respect to (ϕ, α) on an open interval J of $\mathbb{R}$. Then the following estimate holds:*

$$d(u(t), v(t)) \leq e^{-\alpha(t-s)} d(u(s), v(s)) \tag{1.26}$$

for all $s, t \in J$ such that $s < t$.

In the proof of Proposition 1.1 we shall use the following lemma.

Lemma 1.2. *Let J be an open interval of $\mathbb{R}$, $g : J \to \mathbb{R}$ be continuous and $\eta \in C^1(J; \mathbb{R})$ with compact support in J. Let $a, b \in \mathbb{R}$, $a < b$, and $\delta > 0$ be such that*

$$\operatorname{supp} \eta \subseteq [a, b] \subset [a - \delta, b + \delta] \subset J.$$

Then we have

$$-\int_J g(t)\dot{\eta}(t) \, dt = \lim_{\substack{0<h<\delta \\ h\to 0}} \int_a^b \frac{1}{h}\big[g(t + h) - g(t)\big]\eta(t) \, dt. \tag{1.27}$$

Noticing that for $0 < h < \delta$

$$\int_a^b g(t+h)\eta(t)\,dt = \int_{a+h}^{b+h} g(t)\eta(t-h)\,dt = \int_a^b g(t)\eta(t-h)\,dt + \int_b^{b+h} g(t)\eta(t-h)\,dt,$$

we get

$$\int_a^b \frac{1}{h}\big[g(t+h) - g(t)\big]\eta(t)\,dt = -\int_a^b g(t)\frac{1}{h}\big[\eta(t) - \eta(t-h)\big]\,dt + \frac{1}{h}\int_b^{b+h} g(t)\eta(t-h)\,dt.$$

Since, as $h \to 0$, the integrand of the first integral of the RHS converges uniformly to $g(t)\dot\eta(t)$ and the second term tends to $g(b)\eta(b) = 0$, the lemma is proved.

Lemma 1.3. *Let J be an open interval of $\mathbb{R}$, let $f_1 \in C(J;\mathbb{R})$ and $f_2 \in L^1_{\mathrm{loc}}(J)$. Then the following assertions are equivalent.*

(i) $-\int_J f_1(t)\dot\eta(t)\,dt + \int_J f_2(t)\eta(t)\,dt \le 0$ *for every $\eta \in C_c^\infty(J)$ nonnegative.*

(ii) *For every $t_1, t_2 \in J$ such that $t_1 < t_2$*

$$f_1(t_2) - f_1(t_1) + \int_{t_1}^{t_2} f_2(r)\,dr \le 0.$$

Proof. (ii) $\Longrightarrow$ (i).

We apply Lemma 1.2 with $g(t) := f_1(t)$. We obtain

$$\begin{aligned}
-\int_J f_1(t)\dot\eta(t)\,dt &= \lim_{\substack{0<h<\delta\\ h\to 0}} \int_a^b \frac{1}{h}\big[f_1(t+h) - f_1(t)\big]\eta(t)\,dt \\
&\overset{\text{(ii)}}{\le} \overline{\lim_{\substack{0<h<\delta\\ h\to 0}}} \int_a^b \Big(-\frac{1}{h}\int_t^{t+h} f_2(r)\,dr\Big)\eta(t)\,dt = \int_a^b -f_2(t)\eta(t)\,dt
\end{aligned}$$

observing that $\fint_\cdot^{\cdot+h} f_2(r)\,dr \to f_2(\cdot)$ in $L^1(a,b)$ as $h \to 0$.

(i) $\Longrightarrow$ (ii).

Using a standard approximation by means of (Sobolev) mollifiers one can show that the test functions η can be chosen nonnegative absolutely continuous (in particular Lipschitz continuous) with compact support in J. Given $t_1, t_2 \in J$ with $t_1 < t_2$ we choose as test functions η_n for n large enough the linear interpolation of $\eta_n(t_1) = \eta_n(t_2) = 0$ and $\eta_n(t_1 + \frac{1}{n}) = \eta_n(t_2 - \frac{1}{n}) = 1$ on the interval $[t_1, t_2]$ and 0 outside.

We obtain

$$n\int_{t_2-1/n}^{t_2} f_1(t)\,dt - n\int_{t_1}^{t_1+1/n} f_1(t)\,dt + \int_{t_1}^{t_2} f_2(t)\eta_n(t)\,dt \le 0.$$

Taking the limit as $n \to \infty$ we arrive at (ii). □

Remark 1.3. Applying Lemma 1.3 to (EVI) we get

$$\frac{1}{2}d^2(u(t_2),z) - \frac{1}{2}d^2(u(t_1),z) + \frac{\alpha}{2}\int_{t_1}^{t_2} d^2(u(t),z)\,dt \le (t_2 - t_1)\phi(z) - \int_{t_1}^{t_2}\phi(u(t))\,dt \tag{1.28}$$

for every $[t_1, t_2] \subset J$ and $z \in D(\phi)$ (see Remark 4.0.5 of [AGS]).

We shall choose in (EVI) test functions η of the form $\eta(t) := e^{\alpha t}\widetilde{\eta}(t)$ where $\widetilde{\eta} \in C_c^\infty(J)$ nonnegative we obtain

$$-\int_J \frac{1}{2}\, d^2(u(t),z)\big[e^{\alpha t}\big(\dot{\widetilde{\eta}}(t) + \alpha\widetilde{\eta}(t)\big)\big]\, dt \\ + \int_J \frac{\alpha}{2}\, d^2(u(t),z)e^{\alpha t}\widetilde{\eta}(t)\, dt \le \int_J e^{\alpha r}\big(\phi(z) - \phi(u(r))\big)\, dr.$$

So the LHS reduces to

$$-\int_J \frac{1}{2}\, d^2(u(t),z)e^{\alpha t}\dot{\widetilde{\eta}}(t)\, dt$$

and by applying again Lemma 1.3 we obtain

$$\frac{1}{2}\, e^{\alpha t_2} d^2(u(t_2),z) - \frac{1}{2}\, e^{\alpha t_1} d^2(u(t_1),z) \le \left(\int_{t_1}^{t_2} e^{\alpha r}\, dr\right)\phi(z) - \int_{t_1}^{t_2} e^{\alpha r}\phi(u(r))\, dr \tag{1.29}$$

for every $t_1, t_2 \in J$ with $t_1 < t_2$, and any $z \in D(\phi)$.

Now we are in a position to prove Proposition 1.1.

Proof. Suppose u and v are two solutions of (EVI) with respect to (ϕ, α) (gradient flows) on an open interval J of $\mathbb{R}$. Set

$$g_\Delta(t) := \tfrac{1}{2}e^{2\alpha t}d^2(u(t),v(t)), \quad t \in J.$$

Clearly $g_\Delta \in C(J)$. We want to show that g_Δ is nonincreasing on J. In view of Lemma 1.3 with $f := g_\Delta$, $z := 0$, and Lemma 1.2 with $g := g_\Delta$, it is sufficient to prove

$$\varlimsup_{\substack{0<h<\delta \\ h\to 0}} \int_a^b \big[g_\Delta(t+h) - g_\Delta(t)\big]\eta(t)\, dt \le 0$$

for any $\eta \in C_c^\infty(J)$ nonnegative, with a, b, δ as in Lemma 1.2.

We have

$$\begin{aligned} g_\Delta(t+h) - g_\Delta(t) &= e^{2\alpha(t+h)}\tfrac{1}{2}d^2(u(t+h),v(t+h)) - e^{2\alpha t}\tfrac{1}{2}d^2(u(t),v(t)) \\ &= e^{\alpha(t+h)}\Big[e^{\alpha(t+h)}\tfrac{1}{2}d^2(u(t+h),v(t+h)) - e^{\alpha t}\tfrac{1}{2}d^2(u(t),v(t+h))\Big] \\ &\quad + e^{\alpha t}\Big[e^{\alpha(t+h)}\tfrac{1}{2}d^2(u(t),v(t+h)) - e^{\alpha t}\tfrac{1}{2}d^2(u(t),v(t))\Big]. \end{aligned}$$

Using (1.29) both for u and v, we obtain

$$\Big[e^{\alpha(t+h)}\tfrac{1}{2}d^2(u(t+h),v(t+h)) - e^{\alpha t}\tfrac{1}{2}d^2(u(t),v(t+h))\Big] \\ \le \left[\left(\int_t^{t+h} e^{\alpha r}\, dr\right)\cdot \phi\circ v(t+h) - \int_t^{t+h} e^{\alpha r}\phi\circ u(r)\, dr\right]$$

and

$$\Big[e^{\alpha(t+h)}\tfrac{1}{2}d^2(u(t),v(t+h)) - e^{\alpha t}\tfrac{1}{2}d^2(u(t),v(t))\Big] \\ \le \left[\left(\int_t^{t+h} e^{\alpha r}\, dr\right)\cdot \phi\circ u(t) - \int_t^{t+h} e^{\alpha r}\phi\circ v(r)\, dr\right].$$

Hence

$$\begin{aligned}
&\int_a^b \frac{1}{h}\big[g_\Delta(t+h)-g_\Delta(t)\big]\eta(t)\,dt\\
&\quad\le \int_a^b e^{\alpha(t+h)}\left(\frac{1}{h}\int_t^{t+h} e^{\alpha r}\,dr\right)\phi\circ v(t+h)\eta(t)\,dt\\
&\quad- \int_a^b e^{\alpha(t+h)}\left(\frac{1}{h}\int_t^{t+h} e^{\alpha r}\phi\circ u(r)\,dr\right)\eta(t)\,dt\\
&\quad+ \int_a^b e^{\alpha t}\left(\frac{1}{h}\int_t^{t+h} e^{\alpha r}\,dr\right)\phi\circ u(t)\eta(t)\,dt\\
&\quad- \int_a^b e^{\alpha t}\left(\frac{1}{h}\int_t^{t+h} e^{\alpha r}\phi\circ v(r)\,dr\right)\eta(t)\,dt.
\end{aligned}$$

Using $\phi\circ v(\cdot+h)\overset{h\to 0}{\to}\phi\circ v$ in $L^1(a,b)$ as well as $\frac{1}{h}\int_\cdot^{\cdot+h} w(r)\,dr\to w(\cdot)$ in $L^1(a,b)$ for any $w\in L^1_{\text{loc}}(J)$, we obtain

$$\begin{aligned}
&\varlimsup_{\substack{0<h<\delta\\ h\to 0}}\int_a^b \frac{1}{h}\big[g_\Delta(t+h)-g_\Delta(t)\big]\eta(t)\,dt\\
&\qquad\le \int_a^b e^{2\alpha t}\phi\circ v(t)\eta(t)\,dt-\int_a^b e^{2\alpha t}\phi\circ u(t)\eta(t)\,dt\\
&\qquad\qquad+\int_a^b e^{2\alpha t}\phi\circ u(t)\eta(t)\,dt-\int_a^b e^{2\alpha t}\phi\circ v(t)\eta(t)\,dt=0.\ \square
\end{aligned}$$

2 The Hilbert space case

Let $(X,\langle\cdot,\cdot\rangle)$ be a real Hilbert space with corresponding norm $|\cdot|$ and metric $d(\cdot,\cdot)$. Let $e: X\to\mathbb{R}$ be defined by

$$e(x):=\tfrac{1}{2}|x|^2,\quad x\in H. \tag{2.1}$$

Let $\phi: X\to(-\infty,+\infty]$. We recall that ϕ is called proper if $D(\phi):=\{x\in X:\phi(x)<\infty\}$ is not empty. In this case we call ϕ convex if $D(\phi)$ is a convex subset of X and the restriction of ϕ to $D(\phi)$ is convex. We call ϕ *lower semicontinuous* (l.s.c.) if $\{x\in X:\phi(x)\le c\}$ is closed for every $c\in\mathbb{R}$.

We are now in a position to state the main "existence" result in the Hilbert space case when $J:=(0,\infty)$.

Theorem 2.1. *Let $(X,\langle\cdot,\cdot\rangle)$ be a real Hilbert space and let $\phi: X\to(-\infty,+\infty]$ be proper, l.s.c. and such that $\phi-\alpha e$ is convex for some $\alpha\in\mathbb{R}$. Then for every $x\in\overline{D(\phi)}$ there exists a gradient flow $u\colon(0,\infty)\to X$ on (X,d) with respect to the pair (ϕ,α) starting at x.*

Remark 2.1. There are several proofs of Theorem 2.1, but the main ingredient in all these proofs is the m-accretivity in $(H,|\cdot|)$ of the subdifferential of the function $\psi:=\phi-\alpha e$.

We first recall some definitions.

Definition 2.1 (subdifferential)**.** Let $\varphi:(X,\langle\cdot,\cdot\rangle)\to(-\infty,+\infty]$ be proper. Let $x\in D(\varphi)$ and $y\in H$. We say that y is a *subgradient* of φ at x if the following holds

$$\langle y,z-x\rangle+\varphi(x)\le\varphi(z)\quad\forall z\in D(\varphi). \tag{2.2}$$

The collection of *all* subgradients of φ at x, which is denoted by $\partial\varphi(x)$, possibly empty, is called the *subdifferential* of φ at x.

We may consider the map $x \mapsto \partial\varphi(x)$ as a map from $D(\partial\varphi) \subset X \to 2^X$ (the collection of all subsets of X), where

$$D(\partial\varphi) := \{x \in D(\varphi) : \partial\varphi \neq \varnothing\}.$$

The graph of this map is the subset $G(\partial\varphi)$ of $X \times X$ defined by

$$G(\partial\varphi) := \{(x, y) \in D(\varphi) \times X : y \text{ is a subgradient of } \varphi \text{ at } x\}.$$

It is customary to use the same notation for $G(\partial\varphi)$ and $\partial\varphi$.

Definition 2.2. A subset A of $X \times X$ is called *monotone* if for every pair $(x_i, y_i) \in A$, $i = 1, 2$, we have

$$\langle y_2 - y_1, x_2 - x_1\rangle \geq 0, \tag{2.3}$$

and *cyclically monotone* if for every finite cyclic sequence $x_0, x_1, \ldots, x_n = x_0$ in $D(A)$ (i.e., $x \in H$ such that for every $y \in H$ with $(x, y) \in A$) and every sequence $y_1, \ldots, y_n$ with $(x_i, y_i) \in A$, $1 \leq i \leq n$, we have

$$\sum_{i=1}^{n} \langle y_i, x_i - x_{i-1}\rangle \geq 0.$$

Problem 2.1 ([R])**.**

(i) Show that if $\phi : (X, \langle\cdot,\cdot\rangle) \to (-\infty, +\infty]$ is proper, the subdifferential $\partial\varphi$ is cyclically monotone, hence monotone.

(ii) Show that if $A \subset X \times X$ is not empty and cyclically monotone, there exists $\varphi : X \to (-\infty, +\infty]$ proper, l.s.c. and convex such that $A \subseteq \partial\varphi$.

Hint: Take $(x_0, y_0) \in A$ and for any $x \in X$ set

$$\begin{aligned}\phi(x) := \sup\{\langle y_n, x - x_n\rangle + \langle y_{n-1}, x_n - x_{n-1}\rangle + \ldots + \langle y_0, x_1 - x_0\rangle :&\\ (x_i, y_i) \in A,\ i = 1, \ldots, n,\ n \in \mathbb{N}_+\}.&\end{aligned}$$

Definition 2.3. Let $(E, \|\cdot\|)$ be a real Banach space. A subset $A \subset E \times E$ is called *accretive* ($-A$ is called *dissipative*) if

$$\|x_1 - x_2\| \leq \|x_1 - x_2 + h(y_1 - y_2)\| \tag{2.4}$$

for every $(x_i, y_i) \in A$, $i = 1, 2$, and every $h > 0$.

Observe that "accretivity" and "monotonicity" of A are equivalent in a real Hilbert space $(H, \langle\cdot,\cdot\rangle)$. This follows from

$$\langle x, y\rangle \geq 0 \quad \text{iff} \quad |x| \leq |x + ty| \text{ for all } t > 0,$$

for any $x, y \in X$.

Problem 2.2. Let $(E, \|\cdot\|)$ be a real Banach space and let $A \subset E \times E$ be accretive. Show that the two following assertions are equivalent:

(i) $R(I + hA) = E$ for some $h > 0$,

(ii) $R(I + hA) = E$ for all $h > 0$.

Here $R(I + hA)$ denotes the range of the graph $I + hA$, i.e.,

$$R(I + hA) := \{y \in E : \text{there exists } x \in D(A) \text{ such that } (x, y) \in I + hA\}.$$

We also write

$$\{y \in E : \text{there exists } x \in D(A) \text{ such that } y \in x + hAx\}.$$

Definition 2.4. Let $(E, \|\cdot\|)$ be a real Banach space. An accretive graph $A \subset E \times E$ is called m-accretive ($-A$ is called m-dissipative) if $R(I + hA) = E$ for every $h > 0$.

Example 2.1. If $X = \mathbb{R}$ then the graph of the function $x \mapsto \operatorname{sign} x$ is monotone but not m-accretive. However, the graph defined by

$$\begin{aligned} (x, 0) \in A &\quad \text{for every } x < 0 \\ (0, y) \in A &\quad \text{for every } y \in [-1, 1] \\ (x, 1) \in A &\quad \text{for every } x > 0 \end{aligned}$$

is m-accretive.

After these preparations we can formulate an important result of the theory of monotone operators in Hilbert spaces.

Proposition 2.1. *Let $(X, \langle\cdot,\cdot\rangle)$ be a real Hilbert space and let $\varphi : X \to (-\infty, +\infty]$ be proper, l.s.c. and convex. Then the subdifferential $\partial\varphi$ is cyclically monotone and m-accretive in $(X, |\cdot|)$. Conversely, if $A \subset X \times X$ is a cyclically monotone graph which is m-accretive, then there exists a proper, l.s.c. and convex functional $\varphi : X \to (-\infty, +\infty]$ such that $A = \partial\varphi$. Moreover, if φ_1 is as above and satisfies $A = \partial\varphi_1$, then there exists some $c \in \mathbb{R}$ such that $\varphi_1(x) = \varphi(x) + c$ for all $x \in H$.*

The first assertion in the conclusion of Proposition 2.1 will be proved below.

Problem 2.3. Prove the second assertion of Proposition 2.1.

Hint: By Problem 2.1 there exists φ as in Proposition 2.1 such that $A \subseteq \partial\varphi$. Hence $I + A \subseteq I + \partial\varphi$ and $(I + A)^{-1} \subseteq (I + \partial\varphi)^{-1}$. Using the m-accretivity of A show that $D\big((I+A)^{-1}\big) = D\big((I+\partial\varphi)^{-1}\big) = X$. Using the monotonicity of $\partial\varphi$ show that $(I+\partial\varphi)^{-1}$ is the graph of a map from X into X.

Hence $(I + A)^{-1} = (I + \partial\varphi)^{-1}$. Therefore, $I + A = I + \partial\varphi$ from which we conclude $A = \partial\varphi$. This arguments shows that an m-accretive graph is maximal with respect to inclusion for the collection of accretive graphs in $X \times X$.

In order to prove Theorem 2.1 we would like to solve the initial value problem for the differential inclusion

$$-\dot{u}(t) - \alpha u(t) \in \partial\psi(u(t)), \quad t > 0,$$

where $\psi := \phi - \alpha e$; and then proceed as in Section 1.

This can be done in the following way. Setting $A := \partial\psi + \alpha I$ we observe that this operator satisfies the condition of the Crandall-Liggett theorem (See Appendix 4).

Defining

$$I_\alpha := \begin{cases} (0,\infty) & \text{if } \alpha \ge 0, \\ \left(0, \frac{1}{|\alpha|}\right) & \text{if } \alpha < 0 \end{cases} \tag{2.5}$$

or equivalently $I_\alpha = \{h > 0\colon 1 + \alpha h > 0\}$, we have that for $h \in I_\alpha$: $(I + hA)^{-1}$ is the graph of a map from $X \to X$ (everywhere defined) which we shall denote by J_h:

$$J_h x\colon = (I + hA)^{-1}x, \quad x \in H,\ h \in I_\alpha. \tag{2.6}$$

Moreover $[J_h]_{\mathrm{Lip}} \le (1 + \alpha h)^{-1}$.

It follows from Theorem I in [CL71], [Appendix 4] that

$$S(t)x := \lim_{n\to\infty} (J_{t/n})^n x, \quad x \in \overline{D(A)} \tag{2.7}$$

exists for $t > 0$, and $\lim_{t\to 0} S(t)x = x$. We set $S(0)x = x$ for $x \in \overline{D(A)}$. Moreover $\{S(t)\}_{t\ge 0}$ is a one-parameter semigroup of operators on $\overline{D(A)}$.

It follows from Theorem II and Lemma 2.3 of [CL71] that the function $u\colon [0,\infty) \to X$ defined by

$$u(t) := S(t)x, \quad x \in D(A) \tag{2.8}$$

satisfies: for every $T > 0$, $u|_{[0,T]} \in \mathrm{Lip}([0,T];X)$, (hence u is differentiable a.e. since X is reflexive) and for almost all $t \in (0,\infty)$ we have $u(t) \in D(A)$ together with $-\dot{u}(t) \in Au(t)$.

In [Br73] Brezis proved that if moreover $A - \alpha I$ is cyclically monotone, then even for $x \in \overline{D(A)}$, the following holds: $u(t) \in D(A)$ for every $[\varepsilon, T]$ with $\varepsilon > 0$, $u|_{[0,T]} \in \mathrm{Lip}([\varepsilon,T];X)$,$-\dot{u}(t) \in Au(t)$ a.e. in $(0,\infty)$.

Proceeding as in Step 5 of Section 1.1 we can show that u is a gradient flow on X with respect to (ϕ,α), starting at x. The conclusion of Theorem 1.1 follows by observing that $\overline{D(A)} := \overline{D(\partial\psi)} = \overline{D(\psi)} =: \overline{D(\phi)}$.

If we want to use this approach in the case of a metric space, we need to be able to define $J_h x$ *without using the linear structure* of the Hilbert space. This can be done. Setting

$$\Phi(h,x;y) := \begin{cases} \frac{1}{2h} d^2(x,y) + \phi(y), & y \in D(\phi) \\ +\infty, & \text{otherwise,} \end{cases} \tag{2.9}$$

we shall show that if $h \in I_\alpha$ and $x \in X$, then

$$J_h x \text{ is the unique global minimizer of } \Phi(h,x;\cdot). \tag{2.10}$$

More precisely, we have the following lemma.

Lemma 2.1. *Let $\phi\colon X \to (-\infty,+\infty]$ be proper, let $\alpha \in R$, $h \in I_\alpha$ and $\psi := \phi - \alpha e$. Then*

i) if $x_0 \in X$, $x_1 \in D(\partial\psi)$ satisfy

$$-\frac{1}{h}(x_1 - x_0) - \alpha x_1 \in \partial\psi(x_1), \tag{2.11}$$

then

$$\Phi(h,x_0;x_1) \le \Phi(h,x_0;z), \quad \textit{for every } z \in X. \tag{2.12}$$

ii) Conversly, if we suppose in addition that ψ is convex, then for any $x_0 \in X$ and $x_1 \in D(\phi)$ satisfying (2.12), we have $x_1 \in D(\partial\psi)$ and (2.11) holds.

Proof. i)⇒ ii). Notice that $x_1 \in D(\partial\psi) \subset D(\psi)$. We first treat the case $\alpha = 0$ ($\phi = \psi$).

$$\begin{aligned}
&\Phi(h, x_0; z) - \Phi(h, x_0; x_1) \\
=&\frac{1}{2h}|x_0 - z|^2 - \frac{1}{2h}|x_0 - x_1|^2 + \psi(z) - \psi(x_1) \\
\geq&\frac{1}{2h}|x_0 - z|^2 - \frac{1}{2h}|x_0 - x_1|^2 + \frac{1}{h}\langle x_0 - x_1, z - x_1\rangle \\
=&\frac{1}{2h}|x_0 - z|^2 - \frac{1}{2h}|x_0 - x_1|^2 + \frac{1}{h}\langle x_0 - x_1, x_0 - x_1\rangle + \frac{1}{h}\langle x_0 - x_1, z - x_0\rangle \\
=&\frac{1}{2h}\left[|z - x_0|^2 + |x_0 - x_1|^2 + 2\langle x_0 - x_1, z - x_0\rangle\right] \\
=&\frac{1}{2h}\left|(z - x_0) + (x_0 - x_1)\right|^2 \\
=&\frac{1}{2h}d^2(x_1, z)
\end{aligned}$$

which is nonnegative.

In the general case we get an extra term in (2.11) and two extra terms coming from the definition of ψ. We get

$$\begin{aligned}
&\Phi(h, x_0; z) - \Phi(h, x_0; x_1) \\
\geq&\frac{1}{2h}d^2(x_1, z) - \alpha\langle x_1, z - x_1\rangle + \frac{\alpha}{2}|z|^2 - \frac{\alpha}{2}|x_1|^2 \\
=&\frac{1}{2h}d^2(x_1, z) + \alpha 2(|x_1|^2 - 2\langle x_1, z\rangle + |z|^2) \\
=&\frac{1}{2}(\frac{1}{h} + \alpha)d^2(x, z) \geq 0
\end{aligned}$$

since $h \in I_\alpha$.

ii)⇒ i). Let $x_0 \in X$, $x_1 \in D(\phi)$ satisfying (2.12). Let $z \in D(\phi)$ and $t \in (0, 1)$. Then, using the convexity of ψ, we have $(1 - t)x_1 + tz \in D(\psi)$ and

$$t[\psi(z) - \psi(z) \geq \psi((1 - t)x_1 + tz) - \psi(x_1).]$$

Using the definition of Φ and (2.12), we get

$$\begin{aligned}
&\psi((1 - t)x_1 + tz) - \psi(x_1) \\
=&\Phi(h, x_0; (1 - t)x_1 + tz) - \Phi(h, x_0; x_1) \\
&\quad - \frac{1}{2h}\left[|(1 - t)x_1 + tz - x_0|^2 - |x_1 - x_0|^2\right] - \frac{\alpha}{2}\left(|(1 - t)x_1 + tz|^2 - |x_1|^2\right) \\
\geq& - \frac{1}{2h}\left[|(1 - t)x_1 + tz - x_0|^2 - |x_1 - x_0|^2\right] - \frac{\alpha}{2}\left(|(1 - t)x_1 + tz|^2 - |x_1|^2\right).
\end{aligned}$$

Dividing by t and letting t tend to zero we arrive at

$$\psi(z) - \psi(z_1) \geq -\frac{1}{h}\langle z - x_1, x_1 - x_0\rangle - \alpha\langle x_1, z - x_1\rangle$$

which is (2.11). □

Having shown that we can define $J_h x$ in terms of the metric d, the functional ϕ and $\alpha \in \mathbb{R}$, it is important to investigate whether the condition $\phi - \alpha e$ convex can also be expressed in terms of these data. The condition $\phi - \alpha e$ means that for every $y_0, y_1 \in D(\phi)$ and $t \in (0,1)$ we have

$$\begin{aligned}&\phi((1-t)y_0 + ty_1) - \alpha e((1-t)y_0 + ty_1)\\ \leq&(1-t)\phi(y_0) + t\phi(y_1) - \alpha(1-t)e(y_0) - \alpha t e(y_1).\end{aligned}$$

Observe that

$$e((1-t)y_0 + ty_1) = (1-t)e(y_0) + te(y_1) - t(1-t)e(y_0 - y_1) \tag{2.13}$$

holds for every $t \in \mathbb{R}$ and $y_0, y_1 \in X$. Indeed

$$\begin{aligned}&e((1-t)y_0 + ty_1) + t(1-t)e(y_0 - y_1)\\ =&\frac{1}{2}(1-t)^2|y_0|^2 + \frac{1}{2}t^2|y_1|^2 + (1-t)t\langle y_0, y_1\rangle\\ &\qquad + \frac{1}{2}(1-t)^2|y_0|^2 + \frac{1}{2}t^2|y_1|^2 - (1-t)t\langle y_0, y_1\rangle\\ =&(1-t)e(y_0) + te(y_1).\end{aligned}$$

It follows that $\phi - \alpha e$ is convex iff ϕ satisfies for $t \in (0,1)$, $y_0, y_1 \in D(\phi)$,

$$\phi((1-t)y_0 + ty_1) \leq (1-t)\phi(y_0) + t\phi(y_1) - \frac{\alpha}{2}d^2(y_0, y_1). \tag{2.14}$$

In view of (2.13) again the function $\frac{1}{2}d^2(x,\cdot)$ satisfies for $x, y_0, y_1 \in X$ and $t \in \mathbb{R}$:

$$\begin{aligned}\frac{1}{2}d^2(x,(1-t)y_0 + ty_1) &= e((1-t)(y_0 - x) + t(y_1 - x))\\ &= (1-t)e(y_1 - x) + t(y_1 - x) - t(1-t)e(y_1 - y_0).\end{aligned}$$

Therefore, we have

$$\begin{aligned}&\Phi(h, x; (1-t)y_0 + ty_1)\\ =&\frac{1}{2h}d^2(x,(1-t)y_0 + ty_1) + \phi((1-t)y_0 + ty_1)\\ \leq&(1-t)\left[\frac{1}{2h}d^2(x, y_0) + \phi(y_0)\right] + t\left[\frac{1}{2h}d^2(x, y_1) + \phi(y_1)\right] - \frac{1}{2}(\frac{1}{h} + \alpha)t(1-t)d^2(y_0, y_1)\\ =&(1-t)\Phi(h, x; y_0) + t\Phi(h, x; y_1) - \frac{1}{2}(\frac{1}{h} + \alpha)t(1-t)d^2(y_0, y_1)\end{aligned} \tag{2.15}$$

for every $h > 0$, $x \in X$, $y_0, y_1 \in D(\phi)$, $t \in (0,1)$ iff $\phi - \alpha e$ is convex.

Observe that if $h \in I_\alpha$, then the function $\Phi(h, x; \cdot)$ is strictly convex.

We conclude this section by showing that in the Hilbert space case, the functional $\Phi(h, x; \cdot)$ possesses a global minimizer if the following conditions are satisfied:

- ϕ is proper and l.s.c.
- $\exists \alpha \in \mathbb{R}$ such that $\phi - \alpha e$ is convex
- $x \in H$ and $h \in I_\alpha$.

In a first step we shall assume ϕ bounded below. Then clearly $\Phi(h, x; \cdot)$ is also bounded below. Set $\overline{\gamma} := \inf\{\Phi(h, x; y) \colon y \in D(\phi)\}$. Let $\{y_n\} \subset D(\phi)$ be a minimizing sequence i.e. $\lim_{n\to\infty} \Phi(h, x; y_n) = \overline{\gamma}$. We claim that $\{y_n\}$ is a Cauchy-sequence in H. Indeed from (2.15) with $t = \frac{1}{2}$ and $\frac{1}{h} + \alpha > 0$ we obtain for $m, n \geq 1$, noticing that $\frac{1}{2}(y_m + y_n) \in D(\phi)$,

$$\begin{aligned}\frac{1}{2}(\frac{1}{h} + \alpha)d^2(y_m, y_n) \leq& \frac{1}{2}\Phi(h, x; y_m) + \frac{1}{2}\Phi(h, x; y_n) - \frac{1}{2}\Phi(h, x; \frac{y_m + y_n}{2})\\ \leq& \frac{1}{2}[\Phi(h, x; y_m) - \overline{\gamma}] + \frac{1}{2}[\Phi(h, x; y_n) - \overline{\gamma}],\end{aligned}$$

which tends to zero as $m, n \to \infty$.

In view of the *completeness* of (X, d) there exists $\bar{y} \in X$ such that $d(y_n, \bar{y}) \to 0$ as $n \to \infty$. Since ϕ is *l.s.c*, $\Phi(h, x; \cdot)$ is also l.s.c., hence

$$\Phi(h, x; \bar{y}) \leq \varliminf_{n\to\infty} \Phi(h, x; y_n) = \overline{\gamma} < \infty.$$

Therefore $\bar{y} \in D(\phi)$ and $\overline{\gamma} \leq \Phi(h, x; \bar{y}) \leq \overline{\gamma}$ which completes the proof in this case. Now we remove the additional assumption ϕ bounded below by using the fact that a proper, l.s.c convex function in a Hilbert space is bounded below by a continuous affine function of the form $y \mapsto a + \langle b, y\rangle$ where $a \in \mathbb{R}$, $b \in H$. [See Appendix 5]. Then it is easy to verify that $\Phi(h, x; \cdot)$ is bounded below. This implies that J_h (see (2.9)) is well-defined and by Lemma (2.1), $J_h x \in D(\partial\psi)$ and satisfies

$$\frac{1}{h}(x - J_h x) - \alpha J_h x \in \partial\psi(x), \quad h \in I_\alpha. \tag{2.16}$$

In particular, when $\alpha = 0$, (2.16) implies that $R(I + h\partial\psi) = X$ for every $h > 0$, hence $\partial\psi$ is m-accretive in $(X, |\cdot|)$. (first assertion of Proposition 2.1).

Moreover, when α is not necessarily equal to 0, we may define for $h \in I_\alpha$:

$$\phi_h(x) := \Phi(h, x; J_h x) = \frac{1}{2h}|x - J_h x|^2 + \phi(J_h x). \tag{2.17}$$

The function ϕ_h is called the Yosida-Moreau approximation of ϕ. (see Section 3 of [LN1]). (Notice that $\phi_n(x) \neq \phi(J_n x)$ in general (!)).

Problem 2.4. Let $(X, \langle, \rangle)$ and $(E, \langle, \rangle_E)$ be real Hilbert spaces. Let $T \colon D(T) \subset X \to E$ be a closed densely defined operator. Let

$$\phi(x) := \begin{cases} \frac{1}{2}|Tx|_E^2, & x \in D(T), \\ +\infty, & \text{otherwise.} \end{cases}$$

Show that the functional ϕ is quadratic (i.e. $\phi(x+y) + \phi(x-y) = 2(\phi(x) + \phi(y))$, $\forall x, y \in D(\phi)$, and $\phi(\lambda x) = \lambda^2\phi(x)$, $\forall\lambda \in \mathbb{R}$, $\forall x \in D(\phi)$), convex and l.s.c. Let $T^*T \colon D(T^*T) \subset X \to E$ be defined by $D(T^*T) := \{x \in D(T) \colon T^*x \in D(T)\}$ and $T^*Tx := T^*(Tx)$, $x \in D(T^*T)$. Show that $\partial\phi = T^*T$.

3 Semi-discrete approximation

In this section, we shall consider the problem of minimization of the functional $\Phi(h, x; \cdot)$ in the case of a metric space in order to define $J_h x$. Observe that in the Hilbert space case, $J_h x$ has the following interpretation. Let A be the operator defined by $A := \partial\psi + \alpha I$ and let u be a solution to $\dot{u}(t) + Au(t) \ni 0$, $t > 0$ with initial value x. Then $J_h x$ can be interpreted as the approximation of u at h obtained by using the "Euler implicit" scheme $\frac{1}{n}(u(h) - u(0)) + Au(h) \ni 0$. Iterating we obtain a so-called semi-discrete approximation of u: $x, J_h x, (J_h)^2 x, \ldots, (J_h)^n x$. Given $t > 0$ and setting $h := \frac{t}{n}$, the next step consists of proving that $(J_{t/n})^n x$ tends to $u(t)$. In our situation, we shall prove, as in Crandall-Ligett thereom, that $\lim_{n\to\infty}(J_{t/n})^n$ exists and that this limit is a gradient flow starting at x. This will be done in the next sections.

Let (X, d) be a metric space and $\phi\colon X \to (-\infty, +\infty]$ be proper. Motivated by the results of the preceding section, we can formulate the condition introduced in [AGS] which plays the role of the convexity of $\phi - \alpha e$ in the Hilbert case.

(H_1) There exists $\alpha \in \mathbb{R}$ such that for every $x, y_0, y_1 \in D(\phi)$, there exists a map $\gamma\colon [0,1] \to D(\phi)$ satisfying $\gamma(0) = y_0$, $\gamma(1) = y_1$ for which the following inequality holds:

$$\begin{aligned} \frac{1}{2h}d^2(x,\gamma(t)) + \phi(\gamma(t)) \le (1-t)\left[\frac{1}{2h}d^2(x,y_0) + \phi(y_0)\right] \\ + t\left[\frac{1}{2h}d^2(x,y_1) + \phi(y_1)\right] - \left(\frac{1}{h} + \alpha\right)\frac{1}{2}t(1-t)d^2(y_0,y_1), \end{aligned} \tag{3.1}$$

for every t belonging to $(0, 1)$ and every $h > 0$ with $1 + \alpha h > 0$.

Remark 3.1. In (2.16), $\gamma(t) = (1-t)y_0 + ty_1$, $t \in [0,1]$, which is independent of x.

From now on, we shall always assume that the functional ϕ is proper and satisfies (H_1). So when we use $\alpha \in \mathbb{R}$, it will always refer to the α of assumption (H_1). In particular, when we say $h \in I_\alpha$ it means α of (H_1). Given $x \in X$ and $h \in I_\alpha$ we shall define $\Phi(h, x; y)$ as in (2.9) so we have

$$\Phi\colon I_\alpha \times X \times X \to (-\infty, \infty].$$

Lemma 3.1. *Let $\phi\colon X \to (-\infty, \infty]$ be proper and satisfy (H_1). Let $h \in I_\alpha$ and $x \in D(\phi)$. Then*

(i) If $\overline{\gamma} := \inf\{\Phi(h, x; y)\colon y \in X\} > -\infty$, then, for every $y, z \in D(\phi)$,

$$\frac{1}{8}(\frac{1}{h} + \alpha)d^2(y,z) \le \frac{1}{2}(\Phi(h,x;y) - \overline{\gamma}) + \frac{1}{2}(\Phi(h,x;z) - \overline{\gamma}). \tag{3.2}$$

(ii) If $\bar{x} \in D(\phi)$ is a global minimizer of $\Phi(h, x; \cdot)$, then for every $z \in D(\phi)$,

$$\Phi(h,x;z) - \Phi(h,x;\bar{x}) \ge \frac{1}{2}(\frac{1}{h} + \alpha)d^2(\bar{x},z). \tag{3.3}$$

Proof. Notice that $\overline{\gamma} < \infty$ since $\Phi(h, x; \cdot)$ is proper. (i) Use (H_1) with $y_0 := y$, $y_1 := z$, $t = \frac{1}{2}$ and observe $\overline{\gamma} \le \Phi(h, x; \gamma(\frac{1}{2}))$.

(ii) Use (H_1) with $y_0 := z$, $y_1 := \bar{x}$. We obtain

$$\frac{1}{2h}d^2(x,\bar{x}) + \phi(\bar{x}) = \Phi(h,x;\bar{x}) \le \Phi(h,x;\gamma(t))$$

$$(1-t)[\frac{1}{2h}d^2(x,z) + \phi(z)] + t[\frac{1}{2h}d^2(x,\bar{x}) + \phi(\bar{x})] - \frac{1}{2}(\frac{1}{h} + \alpha)t(1-t)d^2(\bar{x},z),$$

$t \in (0,1)$. Hence

$$(1-t)[\frac{1}{2h}d^2(x,\bar{x})+\phi(\bar{x})] \le (1-t)[\frac{1}{2h}d^2(x,z)+\phi(x)] - \frac{1}{2}(\frac{1}{h}+\alpha)td^2(\bar{x},z)].$$

Divding by $1-t$ and letting t tend to 1 we arrive at (3.3). □

Remark 3.2. i) It is obvious that if $\bar{x} \in D(\phi)$ satisfies (3.3) it is a global minimizer of $\Phi(h,x;\cdot)$. Moreover, it follows form (3.3) that $\Phi(h,x;\cdot)$ possesses at most one global minimizer.

ii) The RHS of (3.3) (which would simply be equal to zero without (H_1)) will play an essential role in the proof of existence of a gradient flow).

iii) Noticing that $\frac{1}{h}+\alpha > 0$ in (3.2) we see that any minimizing sequence $\{x_n\} \subset D(\phi)$ is a Cauchy-sequence in (X,d). For the existence theorem, we will assume (X,d) to be *complete* which insures the existence of a limit, denoted by $\bar{x} \in X$. Assuming moreover ϕ, hence also $\Phi(h,x;\bar{x})$ to be l.s.c., we obtain

$$\Phi(h,x;\bar{x}) \le \varliminf_{n\to\infty} \Phi(h,x;x_n) = \overline{\gamma} < \infty.$$

Hence $\bar{x} \in D(\Phi(h,x;\cdot)) = D(\phi)$. Therefore $-\infty < \overline{\gamma} \le \Phi(h,x;\bar{x}) \le \overline{\gamma} < \infty$ which implies that $\bar{x}$ is a (the) global minimizer of $\Phi(h,x;\cdot)$.

In view of the preceding remarks, it follows that under these additional assumptions the boundedness from below of $\Phi(h,x;\cdot)$ implies the existence of a minimizer. Therefore we introduce as in [AGS, (2.4.10)], the next and last assumption on ϕ, namely:

(H_2) There exist $x_* \in D(\phi)$, $r_* > 0$ and $m_* \in \mathbb{R}$ such that $\phi(y) \ge m_*$ for every $y \in X$ satisfying $d(x_*,y) \le r_*$.

Lemma 3.2. *([AGS, Lemma 2.4.8, p. 52]) Let $\phi : X \to (-\infty,+\infty]$ be proper and satisfy (H_1) and (H_2). Let α be as in (H_1) and x_*, r_*, m_* be as in (H_2). Then for every $y \in X$*

$$\text{(3.4)} \qquad \begin{cases} \phi(y) \ge m_* & \text{if } d(x_*,y) \le r_*, \\ \phi(y) \ge c - bd(x_*,y) + \frac{1}{2}\,\alpha d^2(x_*,y) & \text{if } d(x_*,y) > r_*, \end{cases}$$

where $c := \phi(x_*)$ and $b := \frac{1}{r_*}(\phi(x_*) - m_*) - \frac{1}{2}\alpha_+ r_*$ with $\alpha_+ := \max(\alpha, 0)$.

Proof. The first part of (3.4) is simply (H_2). We prove the second part. Assume $y \in D(\phi)$ with $d(x_*,y) > r_*$. From (H_1) with $x := x_*$, $y_0 := x_*$, $y_1 := y$ and $t := \frac{r_*}{d(x_*,y)} \in (0,1)$ we find $y_* := \gamma(t) \in D(\phi)$ independent of $h \in I_\alpha$ such that

$$\text{(3.5)} \quad \frac{1}{2h}\,d^2(x_*,y_*) + \phi(y_*) \le (1-t)\Big[\frac{1}{2h}\,d^2(x_*,x_*) + \phi(x_*)\Big] + t\Big[\frac{1}{2h}\,d^2(x_*,y) + \phi(y)\Big] - \Big(\frac{1}{h}+\alpha\Big)\frac{1}{2}t(1-t)d^2(x_*,y)$$

for every $h \in I_\alpha$.

Multiplying by h (>0) and letting h tend to zero in (3.5) we get

$$\frac{1}{2}\,d^2(x_*,y_*) \le \frac{t^2}{2}\,d^2(x_*,y) = \frac{1}{2}\,r_*^2,$$

hence by (H_2)

$$\phi(y_*) \geq m_*. \tag{3.6}$$

Using (3.6), the nonnegativity of the first term in (3.5) and $d(x_*, x_*) = 0$ we obtain

$$\phi(y) \geq \phi(x_*) - \frac{1}{t}(\phi(x_*) - m_*) - \Big(\frac{1}{h} + \alpha\Big)\frac{t}{2}\, d^2(x_*, y) + \frac{\alpha}{2}\, d^2(x_*, y).$$

In case $\alpha \geq 0$ we let h tend to $+\infty$ and in case $\alpha < 0$ we let h tend to $\frac{1}{|\alpha|}$.

Using the definition of t we obtain (3.4). □

As a simple consequence of Lemma 3.2 we obtain

Corollary 3.1. *Let $\phi : X \to (-\infty, +\infty]$ be as in Lemma 3.2, and $\alpha \in \mathbb{R}$ be as in (H_1). Then for every $h > 0$ satisfying $\frac{1}{h} + \alpha > 0$, for every $\overline{x} \in X$, $M > 0$ there exist $\beta > 0$ and $\delta \in \mathbb{R}$ such that*

$$\Phi(h, x; y) \geq \beta d^2(\overline{x}, y) + \delta \quad \text{for every } x \in X \text{ such that } d(x, \overline{x}) \leq M \text{ and for every } y \in X. \tag{3.7}$$

Proof (sketch). Use

$$d^2(x, y) \geq (1 - \varepsilon^2) d^2(\overline{x}, y) - M^2(1/\varepsilon^2 - 1)$$

and

$$d^2(x_*, y) \leq (1 + \eta^2) d^2(\overline{x}, y) + (1 + 1/\eta^2) d^2(x_*, \overline{x})$$

for $0 < \varepsilon, \eta < 1$. □

Under the assumptions of Corollary 3.1 the function $y \mapsto \Phi(h, x; y)$ is bounded from below. We define $\phi_h(x)$ as its infimum on X.

Definition 3.1. Let ϕ be as in Lemma 3.2, $h + \frac{1}{\alpha} > 0$ with $h > 0$ and α as in (H_1).

$$\phi_h(x) := \inf_{y \in X} \Phi(h, x; y). \tag{3.8}$$

Remark 3.3. 1) ϕ_h is a map from X into $\mathbb{R}$.

2) The notation ϕ_h is consistent with the notation of Section 2. Indeed, in Section 2 $\phi_h(x) := \Phi(h, x; J_h x)$ where $J_h x$ is the unique minimizer of $y \mapsto \Phi(h, x; y)$. In this section the existence and uniqueness of such a minimizer will be obtained only for $x \in \overline{D(\phi)}$.

Lemma 3.3. *Let $\phi : X \to (-\infty, +\infty]$ be proper, l.s.c. and satisfy (H_1), (H_2). Then for every $h \in I_\alpha$ the function $\phi_h : X \to \mathbb{R}$ is continuous and for every $x \in \overline{D(\phi)}$ the function $X \ni y \mapsto \Phi(h, x; y)$ possesses a unique global minimizer element of $D(\phi)$ which we denote by $J_h x$. Moreover the map $\overline{D(\phi)} \ni x \mapsto J_h x \in D(\phi)$ is continuous.*

Proof. 1. *Continuity of ϕ_h.*

Let $x_n, \overline{x} \in X$, $n \geq 1$, be such that $\lim_{n\to\infty} d(x_n, \overline{x}) = 0$. Let $y \in D(\phi)$, then $\phi_h(x_n) \leq \Phi(h, x_n; y)$, $n \geq 1$, hence

$$\overline{\lim_{n\to\infty}}\, \phi_h(x_n) \leq \overline{\lim_{n\to\infty}}\, \Phi(h, x_n; y) = \Phi(h, \overline{x}; y).$$

Taking the infimum over $y \in D(\phi)$ we get

$$\overline{\lim_{n\to\infty}}\, \phi(x_n) \le \phi_h(\overline{x}) < \infty.$$

Let $y_n \in D(\phi)$, $n \ge 1$, be such that

$$\Phi(h, x_n; y_n) \le \phi_h(x_n) + \frac{1}{n}, \quad n \ge 1.$$

In view of Corollary 3.1 there exists $C > 0$ such that $d(\overline{x}, y_n) \le C$, $n \ge 1$.

We have $\phi_h(\overline{x}) \le \Phi(h, \overline{x}; y_n)$, $n \ge 1$, hence

$$\begin{aligned}
\phi_h(\overline{x}) &\le \underline{\lim}_{n\to\infty} \Phi(h, \overline{x}; y_n) = \underline{\lim}_{n\to\infty} \Big\{ \frac{1}{2h}\, d^2(\overline{x}, y_n) - \frac{1}{h}\, d(x_n, \overline{x}) d(\overline{x}, y_n) + \phi(y_n) \Big\} \\
&\qquad \text{(since } d(\overline{x}, y_n) \text{ is bounded)} \\
&= \underline{\lim}_{n\to\infty} \Big\{ \frac{1}{2h} \big(d(\overline{x}, y_n) - d(\overline{x}, x_n) \big)^2 + \phi(y_n) \Big\} \\
&\le \underline{\lim}_{n\to\infty} \Big\{ \frac{1}{2h}\, d^2(x_n, y_n) + \phi(y_n) \Big\} \le \underline{\lim}_{n\to\infty} \phi_h(x_n).
\end{aligned}$$

2. *Global minimizer.*

Let $\overline{x} \in \overline{D(\phi)}$ and let $\{y_n\}_{n\ge1} \subset D(\phi)$ be a minimizing sequence, i.e. $\lim_{n\to\infty} \Phi(h, \overline{x}; y_n) = \phi_h(\overline{x})$. As in the proof in Section 2, in view of the lower semicontinuity of $\Phi(h, \overline{x}; \cdot)$ and the completeness of (X, d), it is sufficient to prove that $(y_n)_{n\ge1}$ is a Cauchy sequence. If $\overline{y}$ denotes the limit, note that $\Phi(h, \overline{x}; \overline{y}) < \infty$ hence $\overline{y} \in D(\phi)$. In order to show that (y_n) is a Cauchy sequence we use assumption (H_1) with $x := x_n$, $y_0 := y_n$, $y_1 := y_m$, $t = \frac{1}{2}$, where $(x_n)_{n\ge1} \subset D(\phi)$ such that $\lim_{n\to\infty} d(x_n, \overline{x}) = 0$. Let $C_1 > 0$ be such that $d(x_n, \overline{x}) \le C_1$, $n \ge 1$. From (H_1) we obtain the existence of $y_{n,m} \in D(\phi)$ satisfying

$$\Phi(h, x_n; y_{n,m}) \le \frac{1}{2}\, \Phi(h, x_n; y_n) + \frac{1}{2}\, \Phi(h, x_n; y_m) - \frac{1}{8}\Big(\frac{1}{h} + \alpha\Big) d^2(y_n, y_m).$$

Since $\Phi(h, x_n; y_{n,m}) \ge \phi_h(x_n)$, we get

$$d^2(y_n, y_m) \le 4\Big(\frac{1}{h} + \alpha\Big)^{-1} \big[\big(\Phi(h, x_n; y_n) - \phi_h(x_n)\big) + \Phi(h, x_n; y_m) - \phi_h(x_n) \big], \tag{3.9}$$

for $m, n \ge 1$.

Next we show that the right-hand side of (3.9) tends to zero as $m, n \to \infty$. By Corollary 3.1 we see that any minimizing sequence is bounded, in particular there exists $C_2 > 0$ such that $d(\overline{x}, y_n) \le C_2$, $n \ge 1$. It follows that

$$\begin{aligned}
|\Phi(h, x_n; y_n) - \Phi(h, \overline{x}; y_n)| &= \frac{1}{2h} |d^2(x_n, y_n) - d^2(\overline{x}, y_n)| \\
&\le \frac{1}{2h}\, d(x_n, \overline{x}) \big(d(x_n, y_n) + d(\overline{x}, y_n) \big) \le \frac{1}{2h} (C_1 + 2C_2) d(x_n, \overline{x}) \to 0
\end{aligned}$$

as $n \to \infty$. In view of the continuity of ϕ_h, we get

$$\begin{aligned}
|\Phi(h, x_n; y_n) - \phi_h(x_n)| \le |\Phi(h, x_n; y_n) - \Phi(h, \overline{x}; y_n)| &+ |\Phi(h, \overline{x}; y_n) - \phi_h(\overline{x})| \\
&+ |\phi_h(\overline{x}) - \phi_h(x_n)| \to 0.
\end{aligned}$$

Finally

$$|\Phi(h,x_m;y_m)-\Phi(h,x_n;y_m)| = \frac{1}{2h}\,|d^2(x_m,y_m)-d^2(x_n,y_m)|$$
$$\le \frac{1}{2h}\,d(x_m,x_n)\cdot 2(C_1+C_2)\to 0 \quad \text{as} \quad m,n\to\infty.$$

Since $|\phi_h(x_n)-\phi_h(x_m)|\to 0$, it follows that the right-hand side of (3.9) tends to zero.

Next we prove the uniqueness of the minimizer. Since every minimizing sequence is a Cauchy sequence it is easy to see that the minimizer is unique (construct a new minimizing sequence from two minimizing sequences (u_n) and (v_n) converging respectively to $\overline{u}$ and $\overline{v}$. Then the new minimizing sequence converges to $\overline{w}=\overline{u}=\overline{v}$). Next we prove the continuity of $x\mapsto J_hx$. Let $x_n, x\in \overline{D(\phi)}$, $n\ge 1$ be such that $\lim_{n\to\infty} d(x_n,x)=0$. We have to show that $\lim_{n\to\infty} J_hx_n = J_hx$. In view of part 2. of the proof of Lemma 3.3 it is sufficient to prove that $\{J_hx_n\}$ is a minimizing sequence for $\Phi(h,x;\cdot)$. Indeed, if it is the case, $\{J_hx_n\}$ is a Cauchy sequence whose limit is the global minimizer of $\Phi(h,x;\cdot)$ which is denoted by J_hx. First we show that $\{J_hx_n\}$ is bounded i.e. there exists $y\in X$ (hence for every $y\in X$) $d(y,J_hx_n)$ is bounded in $\mathbb{R}$. Using (3.3) with $x:=x_n$ and some $\bar z\in D(\phi)$ $(\neq\emptyset)$ we get

$$\text{(3.10)} \qquad \frac{1}{2}\left(\frac{1}{h}+\alpha\right)d^2(J_hx_n,\bar z)\le \frac{1}{2h}d^2(x_n,\bar z)+\phi(\bar z)-\phi_h(x_n), \quad n\ge 1.$$

Since $x_n\to x$ and ϕ_h is continuous, the RHS of (3.10) is bounded, hence $\{J_hx_n\}$ is bounded. Let $\{y_n\}$ be a minimizing sequence for $\Phi(h,x;\cdot)$. We know that $\{y_n\}$ is a Cauchy sequence, hence also a bounded sequence. Now we are ready to show that $\{J_hx_n\}$ is also a minimizing sequence for $\Phi(h,x;\cdot)$. Set $\Phi(h,x;J_hx_n)=I_1^{(n)}+I_2^{(n)}$ where

$$I_1^{(n)} := \frac{1}{2h}d^2(x_n,J_hx_n)+\phi(J_hx_n), \quad I_2^{(n)} := \frac{1}{2h}\big[d^2(x,J_hx_n)-d^2(x_n,J_hx_n)\big].$$

Let $\{y_n\}\subset D(\phi)$ be a minimizing sequence for $\Phi(h,x;\cdot)$ i.e. $\lim_{n\to\infty}\Phi(h,x;y_n)=\bar\gamma := \inf_{y\in X}\Phi(h,x;y)>-\infty$. Then by definition of J_hx_n we have

$$I_1^{(n)}\le \frac{1}{2h}d^2(x_n,y_n)+\phi(y_n)=I_3^{(n)}+I_4^{(n)}$$

where $I_3^{(n)} := \frac{1}{2h}d^2(x,y_n)+\phi(y_n)$ and $I_4^{(n)} := \frac{1}{2h}\big(d^2(x_n,y_n)-d^2(x,y_n)\big)$. Hence $\Phi(h,x;J_hx_n)\le I_3^{(n)}+\big(I_4^{(n)}+I_2^{(n)}\big)$. We claim that $\overline{\lim}_{n\to\infty}\big(I_4^{(n)}+I_2^{(n)}\big)\le 0$. Indeed

$$I_4^{(n)}=\frac{1}{2h}(d(x_n,y_n)-d(x,y_n))(d(x_n,y_n)+d(x,y_n))\le \frac{1}{2h}d(x_n,x)[d(x_n,y_n)+d(x,y_n)].$$

Since $\{y_n\}$ is a minimizing sequence for $\Phi(h,x;\cdot)$, it is bounded (see part 2. of the proof above). Moreover, $\{x_n\}$ is also bounded since $d(x_n,x)\to 0$, hence $[d(x_n,y_n)+d(x,y_n)]$ is bounded and $\overline{\lim}_{n\to\infty} I_4^{(n)}\le 0$. Similarly $I_2^{(n)}\le \frac{1}{2h}d(x,x_n)[d(x,J_hx_n)+d(x_n,J_hx_n)]$ and in view of the boundedness of $\{J_hx_n\}$ we obtain $\overline{\lim}_{n\to\infty} I_2^{(n)}\le 0$. Consequently

$$\overline{\lim_{n\to\infty}}\,(I_4^{(n)}+I_2^{(n)})\le \overline{\lim_{n\to\infty}}\,I_4^{(n)}+\overline{\lim_{n\to\infty}}\,I_2^{(n)}\le 0.$$

Since $\lim_{n\to\infty} I_3^{(n)}=\bar\gamma$, we obtain $\overline{\lim}_{n\to\infty}\Phi(h,x;J_hx_n)\le\bar\gamma\le\inf_{n\ge1}\Phi(h,x;J_hx_n)$. Therefore $\lim_{n\to\infty}\Phi(h,x;J_hx_n)=\bar\gamma$ which completes the proof of the continuity of $y\to J_hy$. $\square$

We summarize the results of this section in

Proposition 3.1. *Assume*

- (X, d) *complete metric space,*
- $\phi\colon X \to (-\infty, +\infty]$ *proper, l.s.c.,*
- (H_1) *and (H2),*
- $x \in \overline{D(\phi)}$ *and* $h > 0$, $1 + \alpha h > 0$.

Then the functional $\Phi(h, x; \cdot)$ *defined in* (2.9) *possesses a unique global minimizer in* X *denoted by* $J_h x$. *Moreover* $J_h x \in D(\phi)$ *and satisfies the variational inequality.*

$$\text{(VI)} \qquad \frac{1}{2}\left(\frac{1}{h} + \alpha\right) d^2(J_h x, z) - \frac{1}{2h} d^2(x, z) + \phi_h(x) \le \phi(z)$$

for every $z \in D(\phi)$, where ϕ_h is defined in (3.8).

Proof. The first assertion follows from Lemma 3.3 and (VI) is a reformulation of (3.3) when $x \in D(\phi)$. When $x \in \overline{D(\phi)}$ we approximate x by a sequence $x_n \in D(\phi)$, replace x by x_n in (VI) and pass to the limit using the continuity of both J_h and ϕ_h. □

4 Existence of solutions to EVI

The aim of this section is to establish the existence of solutions to (EVI) via the semi-discrete approximation in the case $\alpha = 0$. For the general case we refer the reader to [AGS], [LN1] (where Crandall-Liggett type estimates are used involving the local slope of ϕ denoted by $|\partial\phi|(x)$) and [CD2] (where Crandall-Liggett type estimates are used not involving $|\partial\phi|(x)$).

In this section we shall make the following assumptions:

- (X, d) is a complete metric space,
- $\phi\colon X \to (-\infty, +\infty]$ is proper and l.s.c..

(A) There exists $h_0 > 0$ such that for every $h \in (0, h_0]$ and every $x \in \overline{D(\phi)}$, the following variational inequality:

find $y \in D(\phi)$ satisfying

$$\text{(4.1)} \qquad \frac{1}{2h}[d^2(y, z) - d^2(x, z)] + \frac{1}{2h} d^2(y, x) + \phi(y) \le \phi(z)$$

for all $z \in D(\phi)$,

possesses a solution.

Remark 4.1. i) (4.1) is (VI) with $\alpha = 0$. It follows from Proposition 3.1 that if ϕ satisfies (H_1) with $\alpha = 0$ and (H_2), then ϕ satisfies (A).

ii) Clearly if $y \in D(\phi)$ satisfies (4.1), then y is a global minimizer of $\Phi(h, x; \cdot)$. Since $\frac{1}{2h} d^2(y, z) + \Phi(h, x; y) \le \Phi(h, x; z)$ for every $z \in D(\phi)$, this global minimizer is unique (choose $z \in D(\phi)$ global minimizer).

Definition 4.1. Let $h \in (0, h_0]$ and $x \in \overline{D(\phi)}$. We denote by $J_h x$ the unique solution to the variational inequality (4.1) and by J_h the map from $\overline{D(\phi)}$ into $D(\phi)$ defined by $x \mapsto J_h x$. We set $J_0 x := x$ and $J_h^0 x := x$, $J_h^n x := J_h(J_h^{n-1} x)$ for every $x \in \overline{D(\phi)}$ and $n \geq 1$. In particular $J_h^1 = J_h$.

As a consequence of (A) we have:

there exist $x_* \in D(\phi)$, $C_1, C_2 \in \mathbb{R}$ such that

$$\phi(z) \geq C_1 + C_2 d(x_*, z) \tag{4.2}$$

for every $z \in D(\phi)$.

Indeed setting $x = x_* \in D(\phi)$, $h = h_0$ in (4.1) we get

$$\phi(z) \geq \phi(J_{h_0} x_*) + \frac{1}{2h_0} d^2(x_*, J_{h_0} x_*) - \frac{1}{2h_0} \left[d(J_{h_0} x_*, z) + d(x_*, z) \right] d(x_*, J_{h_0} x_*).$$

Then (4.2) follows from the triangle inequality.

"Crandall-Liggett estimates"

Our first goal is to show that the sequence $\{J_{t/m}^m x\}$, $t > 0$, $x \in \overline{D(\phi)}$, $t/m < h_0$ is a Cauchy-sequence in (X, d). We proceed as in [CL71]. This is possible thanks to the following

Lemma 4.1. *Let $x, y \in \overline{D(\phi)}$, $\gamma, \delta > 0$, then*

$$d^2(J_\gamma x, J_\delta y) \leq \frac{\gamma}{\gamma + \delta} d^2(J_\gamma x, y) + \frac{\delta}{\gamma + \delta} d^2(x, J_\delta y). \tag{4.3}$$

Proof. Neglecting the third term in (4.1), which is negative, we get

$$\frac{1}{2\gamma} \left[d^2(J_\gamma x, z) - d^2(x, z) \right] + \phi(J_\gamma x) \leq \phi(z)$$

$$\frac{1}{2\delta} \left[d^2(J_\delta y, \hat{z}) - d^2(y, \hat{z}) \right] + \phi(J_\delta y) \leq \phi(\hat{z})$$

for every $z, \hat{z} \in D(\phi)$. Setting $z := J_\delta y$, $\hat{z} := J_\gamma x$ and adding the two inequalities we obtain

$$\left(\frac{1}{\gamma} + \frac{1}{\delta} \right) \frac{1}{2} d^2(J_\gamma x, J_\delta y) \leq \frac{1}{2\delta} d^2(J_\gamma x, y) + \frac{1}{2\gamma} d^2(x, J_\delta y).$$

Multiplying by $2\frac{\gamma\delta}{\gamma+\delta}$ we arrive at (4.3). □

Remark 4.2. In case $(X, \|\cdot\|)$ is a real Banach space and $A \subset X \times X$ is m-accretive [See Appendix 4], and $J_h := (I + hA)^{-1}$, $h > 0$ we have

$$\|J_\gamma x - J_\delta y\| \leq \frac{\gamma}{\gamma + \delta} \|J_\gamma x - y\| + \frac{\delta}{\gamma + \delta} \|x - J_\delta y\|$$

for every $x, y \in X$ and $\gamma, \delta > 0$. Indeed by accretivity of A we get

$$\begin{aligned} \|J_\gamma x - J_\delta y\| &\leq \left\| J_\gamma x - J_\delta y + \frac{\gamma\delta}{\gamma + \delta} \left[\frac{1}{\gamma}(x - J_\gamma x) - \frac{1}{\delta}(y - J_\delta y) \right] \right\| \\ &\leq \frac{\gamma}{\gamma + \delta} \|J_\gamma x - y\| + \frac{\delta}{\gamma + \delta} \|x - J_\delta y\|, \end{aligned}$$

using the fact that $(J_\gamma x, \frac{1}{\gamma}(x - J_\gamma x)) \in A$ and $(J_\delta x, \frac{1}{\delta}(y - J_\delta y)) \in A$. Next we set for $x \in \overline{D(\phi)}$, $\gamma, \delta > 0$:

$$a_{i,j} := \frac{1}{2} d^2(J_\gamma^i x, J_\delta^j x), \quad i, j \in \mathbb{N}_{\geq 0}. \tag{4.4}$$

Using (4.3) and a double induction in i and j we obtain

$$a_{i,j} \leq \frac{\gamma}{\gamma + \delta} a_{i,j-1} + \frac{\delta}{\gamma + \delta} a_{i-1,j}, \quad i, j \in \mathbb{N}_{>0} \tag{4.5}$$

Remark 4.3. Similarly if

$$b_{i,j} := \|J_\gamma^i x - J_\delta^j x\| \tag{4.6}$$

in Remark 4.2 we arrive at

$$b_{i,j} \leq \frac{\gamma}{\gamma + \delta} b_{i,j-1} + \frac{\delta}{\gamma + \delta} b_{i-1,j}, \quad i, j \geq 1. \tag{4.7}$$

In view of Lemma A3.1 of Appendix 3 we can estimate $b_{i,j}$ in terms of $b_{i,0}$ and $b_{0,j}$. We have

$$\begin{aligned} b_{i,0} = \|J_\gamma^i x - x\| &= \|(J_\gamma^i x - J_\gamma^{i-1} x) + (J_\gamma^{i-1} x - J_\gamma^{i-2} x) + \cdots + J_\gamma x - x\| \\ &\leq \sum_{k=1}^{i} \|J_\gamma^k x - J_\gamma^{k-1} x\|. \end{aligned}$$

Using the fact that $[J_\gamma]_{\text{Lip}} \leq 1$ we obtain $\|J_\gamma^k x - J_\gamma^{k-1} x\| \leq \|J_\gamma x - x\|$ for every $k \geq 1$, hence $b_{i,0} \leq i\|J_\gamma x - x\|$. Next we make the stronger assumption $x \in D(A)$. Noticing again that $(J_\gamma x, \frac{1}{\gamma}(x - J_\gamma x)) \in A$, we obtain by accretivity of A:

$$\|x - J_\gamma x\| \leq \|(x - J_\gamma x) + \gamma[y - \frac{1}{\gamma}(x - J_\gamma x)]\| = \gamma\|y\|$$

for any $y \in Ax$. It follows that

$$|||Ax||| := \sup\left\{\frac{1}{\gamma}\|x - J_\gamma x\| \colon \gamma > 0\right\} \leq \inf\{\|y\| : y \in Ax\} < \infty, \tag{4.8}$$

$$\|x - J_\gamma x\| \leq \gamma |||Ax|||, \tag{4.9}$$

$$b_{i,0} \leq (i\gamma)|||Ax|||, \quad i \geq 1. \tag{4.10}$$

$$b_{0,j} \leq (j\delta)|||Ax|||, \quad j \geq 1. \tag{4.11}$$

Notice that instead of $x \in D(A)$ we could choose $x \in X$ such that $|||Ax||| < \infty$. Applying Lemma A3.1 with $K := |||Ax|||$, $r = 1$ we obtain for $m, n \geq 1$,

$$\left\|J_\gamma^m x - J_\delta^n x\right\| \leq |||Ax|||\sqrt{(m\gamma - n\delta)^2 + \gamma\delta(m + n)}. \tag{4.12}$$

In particular choosing $\gamma := \frac{t}{m}$, $\delta := \frac{t}{n}$, $t > 0$ we see that $\{J_{t/m}^m x\}$ is a Cauchy-sequence. We denote its limit by $u(t)$. Moreover

$$\|u(t) - x\| \leq \left\|u(t) - J_{t/m}^m x\right\| + \left\|J_{t/m}^m x - x\right\| \leq \left\|u - J_{t/m}^m x\right\| + t|||Ax|||$$

for every $m \geq 1$. Hence $\|u(t) - x\| \leq t|\!|\!|Ax|\!|\!|$. In particular $\lim_{t\to 0} u(t) = x$. Finally we obtain some regularity in t by setting $\gamma := \frac{t}{m}$, $\delta := \frac{s}{m}$, $0 < s < t$. We obtain

$$\|u(t) - u(s)\| \leq |\!|\!|Ax|\!|\!||t - s|, \quad s, t > 0$$

and by what preceeds also

$$\|u(t) - u(s)\| \leq |\!|\!|Ax|\!|\!||t - s|, \quad s, t \geq 0. \tag{4.13}$$

Hence $u \in \mathrm{Lip}([0, \infty); X)$ and

$$[u]_{\mathrm{Lip}} \leq |\!|\!|Ax|\!|\!|. \tag{4.14}$$

In order to find estimates for $a_{m,n}$ we proceed as in Remark 4.3, so we need to estimate $a_{i,0}$ and $a_{0,j}$. We have

$$d^2(J_\gamma^i x, x) \leq \left(\sum_{k=1}^{i} d(J_\gamma^k x, J_\gamma^{k-1} x)\right)^2 \leq i \sum_{k=1}^{i} d^2(J_\gamma^k x, J_\gamma^{k-1} x)$$

by the Cauchy-Schwarz inequality. We cannot use as in Remark 4.3 the inequality $[J_h]_{\mathrm{Lip}} \leq 1$. However we have

Lemma 4.2. *Let $x \in D(\phi)$. Then i)*

$$\phi(J_h x) \leq \frac{1}{h} d^2(J_h x, x) + \phi(J_h x) \leq \phi(x) \tag{4.15}$$

for every $h \in (0, h_0)$.

ii) Given $T > 0$ there exists $K := K(x, T) > 0$ such that

$$\frac{1}{2} d^2(J_h^l x, x) \leq K(hl) \tag{4.16}$$

for every $h \in (0, h_0)$ and $l \geq 1$ satisfying

$$lh \leq T. \tag{4.17}$$

Proof. i) (4.15) follows from (4.1) where $z := x, y := J_h x$.

ii) From (4.15) we obtain

$$\begin{aligned} d^2(J_h x, x) &\leq h\left(\phi(x) - \phi(J_h x)\right), \\ d^2(J_h^2 x, J_h x) &\leq h\left(\phi(J_h x) - \phi(J_h^2 x)\right), \\ &\vdots \\ d^2(J_h^l x, J_h^{l-1} x) &\leq h\left(\phi(J_h^{l-1} x) - \phi(J_h^l x)\right), \end{aligned}$$

where $hl \leq T$, $l \geq 1$. Adding we get

$$\sum_{k=1}^{l} d^2(J_h^k x, J_h^{k-1} x) \leq h\left(\phi(x) - \phi(J_h^l x)\right). \tag{4.18}$$

Using (4.2) we have

$$-\phi(J_h^l x) \leq |C_1| + |C_2| d(x_*, x) + |C_2| d(x, J_h^l x). \tag{4.19}$$

We estimate

$$d^2(J_h^l x, x) \le \left(\sum_{k=1}^{l} d(J_h^k x, J_h^{k-1} x) \right)^2 \le l \sum_{k=1}^{l} d^2(J_h^k x, J_h^{k-1} x)$$

by the Cauchy-Schwarz inequality. Using (4.17), (4.18) and (4.19):

$$l \sum_{k=1}^{l} d^2(J_h^k x, J_h^{k-1} x) \le (lh)\,[\phi(x) + |C_1| + |C_2| d(x_*, x)] + \frac{1}{2} C_2^2 T(lh) + \frac{1}{2} d^2(x, J_h^l x),$$

since

$$(lh)|C_2| d(x, J_h^l x) \le (lh) \left[\frac{1}{2lh} d^2(x, J_h^l x) + \frac{lh}{2} |C_2|^2 \right]$$

and $(lh)^2 \le (lh)T,\ l \ge 1$. Hence $\frac{1}{2} d^2(J_h^l x, x) \le K(x,T)(lh)$, where

$$K(x,T) := \phi(x) + |C_1| + |C_2| d(x_*, x) + \frac{1}{2} C_2^2 T. \tag{4.20}$$

□

Combining (4.5), (4.16), (4.17) and Lemma A3.1, we obtain

Lemma 4.3. *Let $x \in D(\phi)$, $T > 0$. Then, for every $t \in [0,T]$, $J_{t/m} x$ is well-defined when $m > \frac{T}{h_0}$, and*

$$\lim_{m \to \infty} J_{t/m}^m x \tag{4.21}$$

exists. Notice that for $t = 0$, $J_{t/m}^m x = x$. Moreover, if $u(t)$ is defined as the limit in (4.21), then the function u is 1/2-Hölder continuous on $[0,T]$ and

$$\phi(u(t)) \le \phi(x) (< \infty) \tag{4.22}$$

for every $t \in (0,T]$.

Proof. i) existence of the limit in (4.21). Let $t \in (0,T]$. Let $m, n > \frac{T}{h_0}$, $\gamma := \frac{t}{m}$, $\delta := \frac{t}{n}$, then $\gamma, \delta \in (0, h_0)$, $J_\gamma^i x$, $J_\delta^j x$, $1 \le i \le m$, $1 \le j \le n$ are well-defined, $\gamma m = \delta n = t \le T$. Hence $\{a_{i,j}\}$ defined in (4.4) satisfy Lemma A3.1 with $K = K(x,T)$ in (4.20) and $r = 1$, in view of (4.5) and Lemma 4.1. Therefore,

$$\frac{1}{2} d^2(J_{t/m}^m x, J_{t/n}^n x) \le Kt \sqrt{\frac{1}{m} + \frac{1}{n}},$$

which implies the limit in (4.21) exists in (4.21) thanks to the completeness of X. Choosing $\gamma := \frac{t}{m}$, $\delta := \frac{s}{m}$ with $0 < s < t \le T$, $m > \frac{T}{h_0}$, we obtain by Lemma A3.1

$$\frac{1}{2} d^2(J_{t/m}^m x, J_{s/m}^m x) \le K \sqrt{(t-s)^2 + 2\frac{ts}{m}},$$

hence

$$d^2(u(t), u(s)) \le 2K|t-s|.$$

Finally from Lemma 4.1, (4.16), (4.17) we obtain $\frac{1}{2} d^2(J_{t/m}^m x, x) \le Kt$ which implies $d^2(u(t), x) \le 2Kt$. Therefore $d(u(t), u(s)) \le \sqrt{2K} |t-s|^{1/2}$ for every $0 \le s < t \le T$. Finally we prove (4.22). In view of (4.15) $\phi(J_{t/m} x) \le \phi(x)$, hence $\phi(J_{t/m}^2 x) \le \phi(J_{t/m} x) \le \phi(x)$, $m > \frac{T}{h_0}$. Since ϕ is l.s.c., we get $\phi(u(t)) \le \underline{\lim}_{m \to \infty} \phi(J_{t/m}^m x) \le \phi(x)$. □

Remark 4.4. It follows from the proof of Lemma 4.3 that $\frac{1}{2}d^2(u(t), J^n_{t/n}x) \le Kt\sqrt{\frac{1}{n}}$. This rate of convergence is not optimal. Indeed in [AGS, Theorem 4.0.4] under assumptions (H_1) with $\alpha = 0$ and (H_2) it is shown that $d^2(u(t), J^n_{t/n}x) \le \frac{t}{n}(\phi(x) - \phi_{t/n}(x))$. For related results, we refer the reader to the interesting paper [NS].

In the next lemma we establish an estimate which is useful for weakening the condition $x \in D(\phi)$.

Lemma 4.4. *Let $x \in \overline{D(\phi)}$, $y \in D(\phi)$ and $T > 0$. Then $J_{t/m}x$ and $J_{t/m}y$ are well-defined for every $t \in (0, T]$ and $m > \frac{T}{h_0}$ and we have*

$$\overline{\lim_{m\to\infty}}\, d(J^m_{t/m}x, J^m_{t/m}y) \le d(x, y). \tag{4.23}$$

Proof. As in the proof of of Lemma 4.3 we see that $J_{t/m}x$, $J_{t/m}y$ are well-defined for $t \in [0, T]$ and $m > \frac{T}{h_0}$. For $t = 0$ (4.23) is trivial. Let $t \in (0, T]$ and $m > \frac{T}{h_0}$. Set $\gamma := \frac{t}{m}$. Deleting the third term in (4.1) we obtain

$$\frac{1}{2\gamma}(d^2(J^k_\gamma y, z) - d^2(J^{k-1}_\gamma y, z)) \le \phi(z) - \phi(J^k_\gamma y),$$

$$\frac{1}{2\gamma}(d^2(J^k_\gamma x, \hat{z}) - d^2(J^{k-1}_\gamma x, \hat{z})) \le \phi(\hat{z}) - \phi(J^k_\gamma x),$$

$k = 1, \dots, m$. Setting $z := J^k_\gamma x$ and $\hat{z} := J^{k-1}_\gamma y$, adding and multiplying by γ, we obtain

$$d^2(J^k_\gamma x, J^k_\gamma y) - d^2(J^{k-1}_\gamma x, J^{k-1}_\gamma y) \le 2\gamma[\phi(J^{k-1}_\gamma y) - \phi(J^k_\gamma y)].$$

Summing over k from 1 to m we get

$$d^2(J^m_\gamma x, J^m_\gamma y) \le d^2(x, y) + 2\gamma[\phi(y) - \phi(J^m_\gamma y)]. \tag{4.24}$$

Next we replace γ by t/m and observe that $\lim_{m\to\infty} J^m_{t/m}y$ exists by Lemma 4.3. In view of the lower semicontinuity of ϕ, $\{\phi(J^m_{t/m}y)\}$ is bounded, therefore taking the $\overline{\lim}_{m\to\infty}$ in (4.24) we arrive at (4.23). □

In the next lemma we assume $x \in \overline{D(\phi)}$.

Lemma 4.5. *Let $x \in \overline{D(\phi)}$ and $T > 0$. Then, for $m > \frac{T}{h_0}$ and $t \in [0, T]$, $J_{t/m}x$ is well-defined and*

$$\lim_{m\to\infty} J^m_{t/m}x \tag{4.25}$$

exists. Moreover, if $u(t)$ is defined as the limit in (4.25), then

$$u \in C([0, T]; X), \tag{4.26}$$

$$u(t) \in D(\phi) \text{ for } t \in (0, T], \tag{4.27}$$

and u satisfies

$$\frac{1}{2}d^2(u(t_2), z) - \frac{1}{2}d^2(u(t_1), z) \le (t_2 - t_1)[\phi(z) - \phi \circ u(t_2)] \tag{4.28}$$

for all $0 < t_1 < t_2$ and all $z \in D(\phi)$.

Proof. Let $x \in \overline{D(\phi)}$, $y \in D(\phi)$, $m, n > \frac{T}{h_0}$. We have

$$d(J^m_{t/m}x, J^n_{t/n}x) \le d(J^m_{t/m}x, J^m_{t/m}y) + d(J^m_{t/m}y, J^n_{t/n}y) + d(J^n_{t/n}y, J^n_{t/n}x).$$

By Lemma 4.3 and 4.4, we obtain

$$0 \le \overline{\lim_{m,n\to\infty}} d(J^m_{t/m}x, J^n_{t/n}x) \le d(x,y) + 0 + d(x,y).$$

Since $y \in D(\phi)$ is arbitrary we get $\lim_{m,n\to\infty} d(J^m_{t/m}y, J^n_{t/n}x) = 0$, which implies (4.25). Set $u(t) := \lim_{m\to\infty} J^m_{t/m}x$, $t \in [0,T]$. Let $0 \le s < t \le T$. Then for $m > \frac{T}{h_0}$:

$$\begin{aligned} d(u(s), u(t)) \le d(u(s), J^m_{s/m}x) + d(J^m_{s/m}x, J^m_{s/m}y) \\ + d(J^m_{s/m}y, J^m_{t/m}y) + d(J^m_{t/m}y, J^m_{t/m}x) + d(J^m_{t/m}x, u(t)). \end{aligned}$$

As above, taking $\overline{\lim}_{m\to\infty}$, we obtain

$$d(u(s), u(t)) \le d(x,y) + \sqrt{2K(y,T)}|t-s|^{1/2} + d(x,y).$$

Given $\varepsilon > 0$, choosing y such that $2d(x,y) < \frac{\varepsilon}{2}$ we find that $d(u(s),u(t)) \le \varepsilon$ if $|t - s| \le \frac{\varepsilon^2}{8K(y,T)}$ whenever $K(y,T) > 0$. This implies (4.26). Next we prove (4.28) for t_1, t_2 rationals. There exists $s > 0$ positive rational and $0 < q < p \in \mathbb{N}_{>0}$ such that $t_1 = qs$ and $t_2 = ps$. Observe that if $k \ge m$

$$(J_{s/k})^{qk}x = (J_{\frac{qs}{qk}})^{qk}x = (J_{\frac{t_1}{qk}})^{qk}x \overset{k\to\infty}{\longrightarrow} u(t_1). \tag{4.29}$$

and

$$(J_{s/k})^{pk}x = (J_{\frac{ps}{pk}})^{pk}x = (J_{\frac{t_2}{pk}})^{pk}x \overset{k\to\infty}{\longrightarrow} u(t_2). \tag{4.30}$$

Set $h := \frac{s}{k}$, $x_l := J^l_h x$, $l \ge 1$, $x_0 := x$. From (4.1) (neglecting the 3^d term), we obtain for $z \in D(\phi)$

$$\frac{1}{2}\big(d^2(x_l, z) - d^2(x_{l-1}, z)\big) + h\phi(x_l) \le h\phi(z). \tag{4.31}$$

Summing (4.31) over l from $l := qk+1$ to $l := pk$ we get

$$\frac{1}{2}\big(d^2(x_{pk}, z) - d^2(x_{qk}, z)\big) + h \sum_{l=qk+1}^{pk} \phi(x_l) \le (t_2 - t_1)\phi(z).$$

By (4.15) and induction we have $\phi(x_l) \ge \phi(x_{pk})$, $qk+1 \le l \le pk$, hence

$$\frac{1}{2}\big(d^2(x_{pk}, z) - d^2(x_{qk}, z)\big) + (t_2 - t_1)\phi(x_{pk}) \le (t_2 - t_1)\phi(z).$$

Using (4.29), (4.30) and the lower semicontinuity of ϕ we arrive at

$$(t_2 - t_1)\phi(u(t_2)) \le (t_2 - t_1)\phi(z) - \frac{1}{2}[d^2(u(t_2), z) - d^2(u(t_1), z)]. \tag{4.32}$$

This implies $\phi \circ u(t_2) < \infty$ and (4.28) for $0 < t_1 < t_2$ rationals. Now let $0 < t_1 < t_2$ with t_1 real numbers and t_2 rational. We approximate t_1 by $t_{1,n} < t_2$, $t_{1,n}$ rationals and obtain (4.32) with $0 < t_1 < t_2$, t_2 rational. Finally approximating t_2 by $t_{2,n} > t_1$ rationals and using again the lower semicontinuity of ϕ we arrive at (4.28) showing that $\phi \circ u(t_2) < \infty$ for any $t_2 > 0$, hence (4.27). □

Finally combining Lemma 4.5 with Proposition 1.1 of [CD1] we arrive at

Theorem 4.1. *Let (X, d) be a complete metric space and let $\phi\colon X \to (-\infty, +\infty]$ be proper and lower semicontinuous. Assume (A) and let $h_0 > 0$ be as in (A). Then for every $t > 0$ the operator $J_{t/m}\colon \overline{D(\phi)} \to D(\phi)$ of Definition 4.1 is well-defined when $m > \frac{t}{h_0}$. Moreover, $\lim_{m\to\infty}(J_{t/m})^m x$ exists for every $x \in \overline{D(\phi)}$. If $u(t)$ is defined as the limit above, then $u \in C((0,\infty); X)$, $\phi \circ u \in L^1_{loc}(0,\infty)$ and u is a solution to (EVI) where $J = (0,\infty)$ and $\alpha = 0$. Moreover, $\lim_{t\to 0} u(t) = x$ and u is the only solution to (EVI) where $J = (0,\infty)$ and $\alpha = 0$ having x as initial value. The following additional properties of u hold.*

$$\phi \circ u(t) < \infty \text{ for every } t > 0, \tag{4.33}$$

$$\phi \circ u\colon (0,\infty) \to \mathbb{R} \text{ is nonincreasing} \tag{4.34}$$

and

$$\phi(x) = \lim_{t\to 0} \phi \circ u(t) \text{ whenever } \phi(x) < \infty. \tag{4.35}$$

The function u satisfies

$$\frac{1}{2}d^2(u(t), z) - \frac{1}{2}d^2(u(s), z) \le (t-s)[\phi(z) - \phi(u(t_2))] \tag{4.36}$$

for every $0 < s < t$, and every $z \in D(\phi)$; Moreover $u|_{[a,b]} \in \mathrm{AC}([a,b]; X)$ for every $0 < a < b$ (hence $[a,b] \ni t \to \frac{1}{2}d^2(u(t), z) \in \mathrm{AC}[a,b]$ for every $z \in D(\phi)$) and for every $z \in D(\phi)$:

$$\frac{d}{dt}\frac{1}{2}d^2(u(t), z) + \phi(u(t)) \le \phi(z), \quad \text{a.e. in } (0,\infty). \tag{4.37}$$

Moreover, for every $h > 0$ the function $v(t) := u(t+h)$, $t > 0$ is a solution to (EVI) with $J = (0,\infty)$ and $\alpha = 0$, having $u(h)$ as initial value. Finally if we set

$$S(t)x := \lim_{m\to\infty} J^m_{t/m} x, \quad t > 0$$
$$S(0)x := x,$$

then $S(t)\colon \overline{D(\phi)} \to D(\phi)$ for every $t > 0$ and

$$\begin{cases} S(t+s) = S(t)S(s), & t, s > 0 \\ S(0) = I_{\overline{D(\phi)}} \end{cases} \tag{4.38}$$

$$d(S(t+h)x, S(t+h)y) \le d(S(t)x, S(t)y) \tag{4.39}$$

for every $t \ge 0$, $h > 0$, and $x, y \in \overline{D(\phi)}$, and finally

$$[0,\infty) \ni t \to S(t)x \in C([0,\infty); X) \tag{4.40}$$

for every $x \in \overline{D(\phi)}$. The family $\{S(t)\}_{t\ge 0}$ of operators on $\overline{D(\phi)}$ is called a C_0-contraction semi-group of operators on $\overline{D(\phi)}$.

Proof. It follows from Lemma 4.5 that $u \in C([0,\infty);X)$, $u(t) \in D(\phi)$ for every $t \in (0,T]$ and u satisfies (4.28) on $(0,T)$ for every $T > 0$. Hence $u \in C((0,\infty);X)$ satisfies assertion (A4) in Proposition 1.1 of [CD1] with $\alpha = 0$. Moreover, $\lim_{t\to 0} u(t) = x$ by Lemma 4.5. Then all conclusions of Theorem 4.1 except (4.35), (4.38) are direct consequences of Proposition 1.1 of these notes and Proposition 1.1 of [CD1]. Concerning (4.35) we have by (4.22), $\phi(u(t)) \le \phi(x)$ for all $t > 0$. Since u is continuous at 0 and ϕ is l.s.c, we have for $t_n \downarrow 0$, $\phi(x) = \phi(u(0)) \le \underline{\lim}_{n\to\infty} \phi \circ u(t_n)$. Hence $\phi(x) = \lim_{n\to\infty} \phi \circ u(t_n)$. Finally we establish (4.38). Let $x \in \overline{D(\phi)}$. Set $u(t) := S(t)x$, $t \ge 0$, then u is a solution to (EVI) with initial value x. As a consequence of Proposition 1.1 of [CD1] u satisfies (4.37). Given $s > 0$ set $v(t) := u(t+s)$. It follows that $v \in C([0,\infty);X)$, $v(0) = u(s) = S(s)x$ and it is easy to see that v satisfies (4.37). Again by Proposition 1.1 of [CD1], v is a solution to (EVI) on $(0,\infty)$ with initial value $v(0) = u(s)$. As a consequence of Proposition 1.1 we have $v(t) = S(t)v(0)$, $t > 0$. Indeed $d(v(t), S(t)v(0)) \le d(v(t'), S(t')v(0))$ for every $0 < t' < t$, hence by taking the limit as $t' \to 0$ we obtain

$$0 \le d(v(t), S(t)v(0)) \le d(v(0), S(0)v(0)) = 0.$$

Therefore $S(t)S(s)x = S(t)u(s) = S(t)v(0) = v(t) = u(t+s) = S(t+s)x$, for $t, s > 0$. Since $x \in \overline{D(\phi)}$ is arbitrary, the first assertion of (4.38) is proved. The second being trivial, the proof of Theorem 4.1 is complete. □

Appendix 1

Definition A1.1. Let (X,d) be a metric space and $a, b \in \mathbb{R}$ with $a < b$. A function $u : [a,b] \to X$ is called *absolutely continuous* on $[a,b]$ if to each $\varepsilon > 0$ there corresponds a $\delta > 0$ such that, for all positive integers n and all families $(a_1,b_1),\dots,(a_n,b_n)$ of disjoint open subintervals of $[a,b]$ of total length at most δ, we have

$$\sum_{k=1}^{n} d(u(a_k), u(b_k)) \le \varepsilon. \tag{A1.1}$$

The collection of all such functions is denoted by $\mathrm{AC}([a,b];X)$.
Observe that $\mathrm{AC}([a,b];X) \subset C([a,b];X)$.

We recall a fundamental result of real analysis.

Theorem A1.1. *i) Let $u \in \mathrm{AC}([a,b]);\mathbb{R})$. Then u is differentiable a.e. in (a,b), $u' \in L^1(a,b)$ and*

$$\int_s^t u'(r)\,dr = u(t) - u(s) \quad \textit{for all} \quad a \le s < t \le b. \tag{A1.2}$$

ii) Let $f \in L^1(a,b)$. Then the function $t \mapsto u(t) = \int_a^t f(r)\,dr$ is absolutely continuous on $[a,b]$ and $u'(t) = f(t)$ a.e. in (a,b).

Remark A1.1. ([ABHN, Corollary 1.2.7]) The following generalization of Theorem A1.1 holds. Let X be a reflexive Banach space (in particular a Hilbert space).

(i) If $u \in \mathrm{AC}([a,b];X)$ then u is strongly differentiable a.e. in (a,b), $u' \in L^1(a,b;X)$ and (A1.2) holds where the integral is a Bochner integral.

(ii) If $f \in L^1(a,b;X)$, $u(t) := \int_a^t f(s)\,ds$, $t \in [a,b]$, then $u \in \mathrm{AC}([a,b];X)$ and $u'(t) = f(t)$ a.e. in (a,b).

The following characterization of absolute continuity will be very useful.

Theorem A1.2. *Let $u : [a,b] \to X$, (X,d) a metric space. Then $u \in \mathrm{AC}([a,b];X)$ iff there exists $m \in L^1(a,b)$, $m \geq 0$, such that*

$$\text{(A1.3)} \qquad d(u(s),u(t)) \leq \int_s^t m(r)\,dr \quad \text{for all} \quad a \leq s < t \leq b.$$

Moreover, if $u \in \mathrm{AC}([a,b];X)$,

$$|\dot{u}|(t) := \lim_{h\to 0} \frac{d(u(t+h),u(t))}{|h|}$$

exists for almost all $t \in (a,b)$, $|\dot{u}| \in L^1(a,b)$,

$$d(u(s),u(t)) \leq \int_s^t |\dot{u}|(r)\,dr, \quad a \leq s \leq t \leq b,$$

and if m satisfies (A1.3)*, then $|\dot{u}|(r) \leq m(r)$ a.e. $|\dot{u}|(t)$ is called the metric derivative of u at t.*

Corollary A1.3. *If $u \in \mathrm{AC}([a,b];X)$, then the function $t \mapsto d^2(u(t),z)$ belongs to $\mathrm{AC}([a,b];\mathbb{R})$ for every $z \in X$.*

Appendix 2

The aim of this Appendix is to recall, mostly without proofs, some results concerning functions of bounded variation.

Let (X,d) be a (not necessarily complete) metric space. Let $a,b \in \mathbb{R}$ with $a < b$ and let $u : [a,b] \to X$. Given a partition π, $a = t_0 < t_1 < \ldots < t_n = b$, let

$$V(\pi;u) := \sum_{i=1}^n d\big(u(t_{i-1}),u(t_i)\big).$$

Then u is said to be of *bounded variation* (with respect to the metric d) if $\sup_\pi V(\pi;u) < \infty$. We denote by $\mathrm{BV}([a,b];X)$ the collection of all X-valued functions which are of bounded variation. We use the notation

$$\text{(A2.1)} \qquad V(u;[a,b]) := \sup_\pi V(\pi;u) \quad \text{over all partitions } \pi \text{ of } [a,b].$$

Clearly if $u \in \mathrm{Lip}([a,b];X)$ then $u \in \mathrm{BV}([a,b];X)$ and $V(u;[a,b]) \leq [u]_{\mathrm{Lip}}(b-a)$. As in the case $X = \mathbb{R}$ one shows that if $u \in \mathrm{BV}([a,b];X)$ and $c \in (a,b)$ then $u|_{[a,c]} \in \mathrm{BV}([a,c];X)$, $u|_{[c,b]} \in \mathrm{BV}([c,b];X)$ and

$$\text{(A2.2)} \qquad V(u;[a,b]) = V\big(u|_{[a,c]};[a,c]\big) + V\big(u|_{[c,b]};[c,b]\big).$$

We shall denote by $V_u(t)$ the real-valued function defined by

$$\text{(A2.3)} \qquad V_u(t) := V(u;[a,t]), \quad t \in [a,b].$$

We have for $a \le s < t \le b$

$$d\big(u(s), u(t)\big) \le V_u(t) - V_u(s) = V(u; [s,t]). \tag{A2.4}$$

The function $V_u(\cdot)$ is nondecreasing and satisfies $V_u(a) = 0$.

Let $v : [a,b] \to X$. If there exists a nondecreasing function $M : [a,b] \to \mathbb{R}$ such that

$$d\big(v(s), v(t)\big) \le M(t) - M(s)$$

holds for all $a \le s < t \le b$, then $v \in \mathrm{BV}([a,b]; X)$ and $V_v(t) \le M(t) - M(a)$, $t \in [a,b]$.

It follows from (A2.4) that if $u \in \mathrm{BV}([a,b]; X)$, then the set where u is not continuous is at most countable. Also if $V_u(\cdot)$ is continuous then clearly u is continuous. On the other hand, it can be shown as in the case $X = \mathbb{R}$ that if u is right (resp. left) continuous at $t \in [a,b]$ then $V_u(\cdot)$ is also right (resp. left) continuous at t.

The next lemma is useful.

Lemma A2.1 ([Br73], Appendix)**.** *Let $u \in \mathrm{BV}([a,b]; X)$. Then we have for all h in $(0, b-a)$*

$$\int_a^{b-h} d\big(u(t), u(t+h)\big)\, dt \le hV(u; [a,b]). \tag{A2.5}$$

Proof. Since the set of discontinuity of u is at most countable, the same holds for the bounded functions $t \mapsto d(u(t), u(t+h))$, $t \mapsto V_u(t)$ and $t \mapsto V_u(t+h)$ on $[a, b-h]$. Hence these functions are integrable. Using (A2.4) we have

$$\begin{aligned}\int_a^{b-h} d\big(u(t), u(t+h)\big)\, dt &\le \int_a^{b-h} V_u(t+h) - V_u(t)\, dt \\ &= \int_{a+h}^{b} V_u(t)\, dt - \int_a^{b-h} V_u(t)\, dt \le \int_{b-h}^{b} V_u(t)\, dt \le hV_u(b) = hV(u; [a,b]). \quad \square\end{aligned}$$

A function $u \in C([a,b]; X)$ is not necessarily of bounded variation but if u is absolutely continuous, then it is of bounded variation and $V_u(\cdot) \in \mathrm{AC}[a,b]$ as in the case $X = \mathbb{R}$. Conversely, if $u \in \mathrm{BV}([a,b]; X)$ and $V_u(\cdot) \in \mathrm{AC}[a,b]$ then $u \in \mathrm{AC}([a,b]; X)$.

Let $v : [a,b] \to X$ be such that there exists a function $M : [a,b] \to X$ nondecreasing and absolutely continuous. Then by what precedes we have $v \in \mathrm{BV}([a,b]; X)$ and $V_v(t) \le M(t) - M(a)$, $t \in [a,b]$. It is easy to verify that $V_v(\cdot) \in \mathrm{AC}[a,b]$ hence $v \in \mathrm{AC}([a,b]; X)$. Notice that M is absolutely continuous iff there exists $m \in L^1(a,b)$ nonnegative such that $M(t) - M(s) = \int_s^t m(r)\, dr$, $a \le s < t \le b$. It follows that for $v : [a,b] \to X$ we have $v \in \mathrm{AC}([a,b]; X)$ iff there exists $m \in L^1(a,b)$ nonnegative such that

$$d\big(v(s), v(t)\big) \le \int_s^t m(r)\, dr, \quad a \le s < t \le b. \tag{A2.6}$$

In this case (A2.6) implies $V_v(t) - V_v(s) \le \int_s^t m(r)\, dr$, hence

$$\int_s^t \frac{d}{dr} V_v(r)\, dr \le \int_s^t m(r)\, dr, \quad a \le s < t \le b.$$

It follows that $\frac{d}{dr} V_v(r) \le m(r)$ a.e. in (a,b).

We conclude this Appendix by showing that if $u \in \mathrm{AC}([a,b]; X)$, then the metric derivative $|\dot{u}|(t)$ (see Theorem A1.2) exists for almost all $t \in (a,b)$, $|\dot{u}| \in L^1(a,b)$ and

$$|\dot{u}|(t) = \frac{d}{dt} V_u(t) \quad \text{a.e. in } (a,b).$$

Proof ([AGS], Theorem 1.1.2). Let $u \in \mathrm{AC}([a,b];X)$ and let N_u be a subset of (a,b) with measure zero such that $\frac{d}{dt}V_u(t)$ exists for every $t \in (a,b) \setminus N_u$. Since $u([a,b])$ is compact in X, it is separable. There exists a countable subset E of $u([a,b])$ which is dense in $u([a,b])$. For every $e \in E$ the functions $d(e,u(\cdot)) \in \mathrm{AC}[a,b]$ and let N_e be a subset of (a,b) with measure zero such that $\frac{d}{dt}d(e,u(t))$ exists for every $t \in (a,b) \setminus N_e$.

Set $N := N_u \cup \bigcup_{e\in E} N_e$. For $t \in (a,b) \setminus N$ set

$$\ell(t) := \sup_{e\in E}\Big|\frac{d}{dt}\,d(e,u(t))\Big| \quad \text{and} \quad \ell(t) = 0,\ t \in N.$$

Then ℓ is nonnegative and measurable. We have

$$d\big(u(s),u(t)\big) = \sup_{e\in E}\big|d(e,u(s)) - d(e,u(t))\big| \le \int_s^t \ell(r)\,dr, \quad a \le s < t \le b.$$

Let $t \in (a,b) \setminus N$. Then

$$\ell(t) = \sup_{e\in E}\varliminf_{s\to t}\frac{\big|d(e,u(t)) - d(e,u(s))\big|}{|t-s|} \le \varliminf_{s\to t}\frac{\big|d(u(t),u(s))\big|}{|t-s|}$$
$$\le \varliminf_{s\to t}\frac{|V_u(t) - V_u(s)|}{|t-s|} = \frac{d}{dt}V_u(t).$$

It follows that $\ell \in L^1(a,b)$. Let N_ℓ be a subset of (a,b) of measure zero such that every $t \in (a,b) \setminus N_\ell$ is a Lebesgue point of ℓ. For every $t \in (a,b) \setminus N_\ell$ we have

$$\varlimsup_{s\to t}\frac{d(u(s),u(t))}{|t-s|} \le \ell(t).$$

Hence for every $t \in (a,b) \setminus (N \cup N_\ell)$ we have

$$\varlimsup_{s\to t}\frac{d(u(s),d(u(t))}{|t-s|} \le \varliminf_{s\to t}\frac{d(u(s),d(u(t))}{|t-s|} \le \frac{d}{dt}V_u(t).$$

Therefore on this set the metric derivative $|\dot u|(t)$ exists and $|\dot u|(t) = \ell(t) \le \frac{d}{dt}V_u(t)$.

On the other hand, since $d(u(s),u(t)) \le \int_s^t \ell(r)\,dr$, $a \le t < s \le b$, we have $\frac{d}{dt}V_u(t) \le \ell(t)$ a.e. in (a,b). It follows that $|\dot u|(t) = \frac{d}{dt}V_u(t)$ a.e. in (a,b). □

Appendix 3

The purpose of this Appendix is to state and prove a lemma which is used in the proof of Remark 4.3 and Lemma 4.3. It is a (symmetric) variant of a lemma due to Crandall–Liggett [CL71].

Lemma A3.1. *Let r, γ, δ, K be real numbers satisfying*

$$0 < r \le 2, \quad \gamma, \delta, K > 0. \tag{A3.1}$$

Let m, n be positive integers. Let $\{a_{i,j}\}_{\substack{0\le i\le m\\ 0\le j\le n}}$ be nonnegative *real numbers satisfying*

$$a_{i,j} \le \frac{\gamma}{\gamma+\delta}\,a_{i,j-1} + \frac{\delta}{\gamma+\delta}\,a_{i-1,j}, \quad 1 \le i \le m,\ 1 \le j \le n, \tag{A3.2}$$

$$a_{i,0} \le K(i\gamma)^r, \quad 1 \le i \le m, \tag{A3.3}$$

$$a_{0,j} \le K(j\delta)^r, \quad 1 \le j \le n. \tag{A3.4}$$

Then for $1 \le i \le m$ *and* $1 \le j \le n$

$$a_{i,j} \le K\big[(i\gamma - j\delta)^2 + \gamma\delta(i+j)\big]^{r/2}. \tag{A3.5}$$

Proof. Case $r = 2$. Let $c_{i,j} := (i\gamma - j\delta)^2 + \gamma\delta(i+j)$, $0 \le i \le m$, $0 \le j \le n$. Then $c_{i,j}$ satisfies (A3.2) with *equality*. Indeed,

$$\begin{aligned}
&\frac{\gamma}{\gamma+\delta}\, c_{i,j-1} + \frac{\delta}{\gamma+\delta}\, c_{i-1,j} \\
=&\frac{\gamma}{\gamma+\delta}\big[((i\gamma - j\delta)+\delta)^2 + \gamma\delta(i+j) - \gamma\delta\big] + \frac{\delta}{\gamma+\delta}\big[((i\gamma - j\delta)-\gamma)^2 + \gamma\delta(i+j) - \gamma\delta\big] \\
=&\left(\frac{\gamma}{\gamma+\delta} + \frac{\delta}{\gamma+\delta}\right)\big[(i\gamma - j\delta)^2 + \gamma\delta(i+j) - \gamma\delta\big] \\
&\qquad + \frac{2\gamma\delta}{\gamma+\delta}(i\gamma - j\delta) - \frac{2\gamma\delta}{\gamma+\delta}(i\gamma - j\delta) + \frac{\gamma\delta^2}{\gamma+\delta} + \frac{\gamma^2\delta}{\gamma+\delta} \\
=&c_{i,j} - \gamma\delta + \frac{\delta+\gamma}{\gamma+\delta}\gamma\delta = c_{i,j}.
\end{aligned}$$

Therefore $a_{i,j} - Kc_{i,j}$ satisfies (A3.2).

Moreover,

$$a_{i,0} - Kc_{i,0} \le K(i\gamma)^2 - K(i\gamma)^2 - Ki\gamma\delta \le 0, \quad 1 \le i \le m,$$

$$a_{0,j} - Kc_{0,j} \le K(j\delta)^2 - K(j\delta)^2 - Kj\gamma\delta \le 0, \quad 1 \le j \le n.$$

An easy double induction argument for i and j implies that

$$a_{i,j} - Kc_{i,j} \le 0, \quad 1 \le i \le m, 1 \le j \le n$$

which is (A3.5).

Case $0 < r < 2$. Set $b_{i,j} = (a_{i,j})^{2/r}$. Since $2/r > 1$ we have

$$b_{i,j} = (a_{i,j})^{2/r} \le \Big(\frac{\gamma}{\gamma+\delta}\, a_{i,j-1} + \frac{\delta}{\gamma+\delta}\, a_{i-1,j}\Big)^{2/r} \le \frac{\gamma}{\gamma+\delta}\, b_{i,j-1} + \frac{\delta}{\gamma+\delta}\, b_{i-1,j}$$

by Jensen's inequality. Moreover

$$b_{i,0} \le K^{2/r}(i\gamma)^2, \quad 1 \le i \le m, \quad \text{and} \quad b_{0,j} \le K^{2/r}(j\delta)^2, \quad 1 \le j \le n.$$

Since $a_{i,j} = (b_{i,j})^{r/2}$, the result follows from case $r = 2$. □

Appendix 4

The aim of this Appendix is to state without proofs some results of the theory of "nonlinear semigroups" on Banach and Hilbert spaces.

Notation

Let X be a nonempty set and let $A, B \subset X \times X$.

$$
\begin{aligned}
D(A) &:= \{x \in X : \exists y \in X \text{ such that } (x,y) \in A\} \\
R(A) &:= \{y \in X : \exists x \in X \text{ such that } (x,y) \in A\} \\
A^{-1} &:= \{(y,x) \in X \times X : (x,y) \in A\} \\
I &:= \{(x,x) \in X \times X : x \in X\} \\
A \circ B &:= \{(x,y) \in X \times X : \exists z \in X \text{ with } (x,z) \in B \text{ and } (z,y) \in A)\}
\end{aligned}
$$

Let X be a *real vector space.* If $A, B \subset X \times X$, and $\lambda \in \mathbb{R}$, one sets

$$
\begin{aligned}
A \pm B &:= \{(x, y \pm z) : (x,y) \in A,\ (x,z) \in B\} \\
\lambda A &:= \{(x, \lambda y) : (x,y) \in A\}.
\end{aligned}
$$

Let $(X, \|\cdot\|)$ be a normed space.

Definition A4.1. A nonempty subset B of $X \times X$ is called *accretive* ($-B$ *dissipative*) if, for every $\lambda > 0$,

$$(I + \lambda B)^{-1} : R(I + \lambda B) \to X$$

is single-valued (i.e. $(I + \lambda B)^{-1}x$ is a singleton for every $x \in R(I + \lambda B)$ or, equivalently, $(I + \lambda B)^{-1}$ is the graph of a function from $R(I + \lambda B)$ into X. By abuse of notation we shall also denote the element of this singleton by $(I + \lambda B)^{-1}x$), and we have

$$\|(I + \lambda B)^{-1}x_1 - (I + \lambda B)^{-1}x_2\| \le \|x_1 - x_2\|$$

for every $x_1, x_2 \in R(I + \lambda B)$.

Remark A4.1. Clearly a nonempty set $B \subset X \times X$ is accretive iff

$$\|x_1 - x_2\| \le \|(x_1 - x_2) + \lambda(y_1 - y_2)\|$$

for every $\lambda > 0$ and every $(x_i, y_i) \in B$, $i = 1, 2$.

Remark A4.2. If B is accretive then $\lambda B + \mu I$ is also accretive for $\lambda, \mu \ge 0$. In particular, if $A \subset X \times X$ is such that $A + \omega I$ is accretive for some $\omega \in \mathbb{R}$, then $(I + \lambda A)^{-1}$ is the graph of a function whenever $\lambda > 0$ satisfies $\omega\lambda < 1$.

Theorem A4.2 ([CL71])**.** *Let $(X, \|\cdot\|)$ be a real Banach space and let $A \subset X \times X$ be such that there exists $\omega \in \mathbb{R}$ for which $A + \omega I$ is accretive. Suppose that there exists $\lambda_0 > 0$ such that*

$$R(I + \lambda A) \supseteq \overline{D(A)} \tag{A4.1}$$

for all $\lambda \in (0, \lambda_0)$, where $\overline{D(A)}$ denotes the closure of $D(A)$ in $(X, \|\cdot\|)$.

Then

$$\lim_{n\to\infty} \left[\left(I + \tfrac{t}{n}A\right)^{-1}\right]^n x \tag{A4.2}$$

exists for $x \in \overline{D(A)}$ and $t > 0$.

Let $S(0)x = x$ and $S(t)x$ be the limit in (A4.2) *for $x \in \overline{D(A)}$ and $t > 0$. Then $\{S(t)\}_{t\ge0}$ is a C_0-semigroup on $\overline{D(A)}$ which satisfies $[S(t)]_{\mathrm{Lip}} \le e^{\omega t}$, $t \ge 0$. Moreover, if $x \in D(A)$ and $u(t) := S(t)x$ for $t \ge 0$, then $u|_{[0,T]} \in \mathrm{Lip}([0,T]; X)$ for every $T > 0$.*

Next (for simplicity) we suppose in addition that the set $A \subset X \times X$ satisfies instead of (A4.1) the stronger assumption

$$R(I + \lambda A) = X \tag{A4.3}$$

for all $\lambda > 0$ such that $\omega\lambda < 1$. Then the following holds:

(i) If u defined above is strongly right-differentiable at some $t \in [0, \infty)$, then

$$u(t) \in D(A) \quad \text{and} \quad -\frac{d^+}{dt}u(t) \in Au(t). \tag{A4.4}$$

(ii) If $v \in C([0,T]; X)$ for some $T > 0$ satisfies

(A4.5) $\quad v(0) \in D(A)$,

(A4.6) $\quad v \in \mathrm{AC}([\varepsilon, T]; X) \quad$ for every $\quad \varepsilon \in (0, T)$,

(A4.7) $\quad v$ is strongly differentiable a.e. in $(0, T)$,

(A4.8) $\quad v(t) \in D(A)$ a.e. in $(0, T)$,

(A4.9) $\quad -\dfrac{d}{dt}v(t) \in Av(t)$ a.e. in $(0, T)$,

then

$$v(t) = S(t)v(0) \quad \text{for every} \quad t \in (0, T].$$

Hilbert space case (see [Br73] and references)

If $A + \omega I$ is accretive for some $\omega \in \mathbb{R}$, if assumption (A4.3) holds and $x \in D(A)$, then $t \mapsto S(t)x$ is right-differentiable for every $t \geq 0$.

If moreover $A + \omega I$ is the subdifferential of a proper, lower semicontinuous, convex function $\phi : X \to (-\infty, +\infty]$ and $x \in \overline{D(A)}$ then $t \mapsto S(t)x$ is right-differentiable for every $t > 0$.

Appendix 5

Lemma A5.1. *Let $\psi : X \to (-\infty, +\infty]$ be proper, l.s.c. and convex. Then there exist $b \in X$ and $c \in \mathbb{R}$ such that*

$$\psi(x) \geq \langle b, x \rangle + c, \quad x \in X. \tag{A5.1}$$

Proof. Consider the epigraph of ψ defined by $\mathrm{epi}(\psi) := \{(x, t) \in X \times \mathbb{R} : \psi(x) \leq t\}$. Since ψ is proper, $\mathrm{epi}(\psi) \neq \varnothing$. Moreover, $\mathrm{epi}(\psi)$ is convex in view of the convexity of ψ. We introduce the innerproduct $\langle\langle \cdot, \cdot \rangle\rangle$ in $X \times \mathbb{R}$ defined by $\langle\langle (x_1, t_1), (x_2, t_2) \rangle\rangle := \langle x_1, x_2 \rangle + t_1 t_2$. Clearly $(X \times \mathbb{R}, \langle\langle \cdot, \cdot \rangle\rangle)$ is a Hilbert space. The subset $\mathrm{epi}(\psi)$ is closed in $X \times \mathbb{R}$ as a consequence of the lower semicontinuity of ψ. Let $x_0 \in D(\psi)$ and $t_0 < \psi(x_0)$. Then $(x_0, t_0) \notin \mathrm{epi}(\psi)$. By the projection theorem on closed convex sets in Hilbert spaces, there exists a unique element $(\overline{x}, \overline{t}) \in \mathrm{epi}(\psi)$ satisfying

$$\langle x - \overline{x}, x_0 - \overline{x} \rangle + (t - \overline{t})(t_0 - \overline{t}) \leq 0 \tag{A5.2}$$

for every $(x, t) \in \mathrm{epi}(\psi)$.

Choose $x = x_0$ and $t \geq \phi(x_0)$ in (A5.2). Since $0 < \langle x_0 - \overline{x}, x_0 - \overline{x} \rangle$ we see that $t_0 - \overline{t}$ cannot be zero. Moreover, choosing $t > \overline{t}$ shows that $t_0 - \overline{t}$ has to be negative. Finally, choosing $x \in D(\psi)$ and $t = \psi(x)$ in (A5.2) we obtain (A5.1) with

$$b := \frac{1}{\overline{t} - t_0}(\overline{x} - x_0) \quad \text{and} \quad c := \overline{t} + \frac{1}{\overline{t} - t_0} \langle \overline{x}, \overline{x} - x_0 \rangle .$$

Clearly (A5.1) holds for $x \in X \setminus D(\psi)$. □

References

[AGS] L. Ambrosio, N. Gigli, G. Savaré, *Gradient Flows in Metric Spaces and in the Space of Probability Measures*, Lectures in Mathematics, ETH Zürich, Birkhäuser Verlag, Basel 2005.

[ABHN] W. Arendt, C. J. K. Batty, M. Hieber, F. Neubrander, *Vector-valued Laplace Transforms and Cauchy Problems*, Monographs in Mathematics, vol. 96, Birkhäuser, Basel, 2001.

[Br73] H. Brezis, *Opérateurs maximaux monotones et semi-groupes de contractions dans les espaces de Hilbert*, North-Holland Mathematics Studies No. 5, Notas de Matemática (50), North-Holland Publishing Co., Amsterdam 1973.

[CD1] Ph. Clément, W. Desch, *Some Remarks on the Equivalence between Metric Formulations of Gradient Flows*, Bollettino U.M.I. (9), III (2010), 583–588 .

[CD2] Ph. Clément, W. Desch, *A "Crandall–Liggett" approach to "Gradient flows" in metric spaces*, submitted.

[LN1] Ph. Clément, *An Introduction to Gradient Flows in Metric Spaces*, Report of the Mathematical Institute Leiden, MI–2009–09, University of Leiden, 2009.

[CL71] M. G. Crandall, T. M. Liggett, *Generation of semi-groups of nonlinear transformations on general Banach spaces*, Amer. J. Math. 93 (1971), pp. 265–298.

[K] T. Kato, *Perturbation theory for linear operators*, Springer-Verlag, 1976.

[NS] R. H. Nochetto, G. Savaré, Nonlinear evolution governed by accretive operators in Banach spaces: error control and applications, Math. Models Methods Appl. Sci. 16 (2006), no. 3, pp. 439–477.

[R] R. T. Rockafellar, *Characterization of the subdifferentials of convex functions*, Pacific J. Math. 17 (1966), pp. 497–510.