Evolution Variational Inequalities in Probability Spaces: An Introduction (III)

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Preface

The aim of these Lecture Notes is to provide an introduction to the theory of gradient flows in probability spaces developed in the second part of the book by Ambrosio–Gigli–Savaré [2]. The main objective is to show that the solution to the heat equation on $\mathbb{R}^d$, having a Borel probability measure with finite second moment as initial value, satisfies a certain evolution variational inequality (EVI) introduced in the first part of [2]. To be more precise, let (X, d) be a complete metric space, let $\phi : X \to (-\infty, +\infty]$ be not identically $+\infty$ and lower semicontinuous, and let $\alpha \in \mathbb{R}$. A continuous function $u : (0, \infty) \to X$ is called a solution to (EVI) if it satisfies

(i) $\phi(u(t)) < \infty$ for every $t > 0$,

(ii) $\frac{1}{2}\mathrm{e}^{\alpha t_2}\,(d(u(t_2), z))^2 - \frac{1}{2}\mathrm{e}^{\alpha t_1}\,(d(u(t_1), z))^2 \le \big(\int_{t_1}^{t_2} \mathrm{e}^{\alpha t}\,dt\big)(\phi(z) - \phi(u(t_2)))$ for every $0 < t_1 < t_2$ and every $z \in X$ such that $\phi(z) < \infty$.

Equivalent formulations can be found in [2], [12] and [11]. It can be shown that there exists at most one solution to (EVI) satisfying a prescribed initial value u_0, i.e. $\lim\limits_{t\to 0} u(t) = u_0$ in x. It is relatively easy to show that the solution to the heat equation on $\mathbb{R}^d$ with initial value $f \in L^2(\mathbb{R}^d)$ given by

$$u(t)(x) := \int_{\mathbb{R}^d} \frac{1}{(4\pi t)^{d/2}}\,\mathrm{e}^{-|x-y|_2^2/(4t)}\,f(y)\,dy,$$

for $t > 0$ and $x \in \mathbb{R}^d$, satisfies (EVI) with $X = L^2(\mathbb{R}^d)$ equipped with its usual metric, ϕ the Dirichlet functional, i.e.

$$\phi(g) := \begin{cases} \frac{1}{2}\int_{\mathbb{R}^d} |\nabla g|_2^2\,dx & \text{when } g \in W^{1,2}(\mathbb{R}^d), \\ +\infty & \text{otherwise} \end{cases}$$

and $\alpha = 0$ (see e.g. [8]). In their celebrated paper Jordan, Kinderlehrer and Otto [16] showed that the Fokker–Planck equation, in particular the heat equation, can be interpreted as a gradient flow in the space $\mathcal{P}_2(\mathbb{R}^d)$, the space of Borel probability measures on $\mathbb{R}^d$ with finite second moment, equipped with the L_2-Wasserstein metric $W_2(\cdot,\cdot)$, for a certain functional ϕ. In the case of the heat equation $\partial_t \rho_t(x) = \Delta \rho_t(x)$, they showed that this functional is the negative of the Gibbs–Boltzmann entropy functional

$$\text{(GBE)} \qquad \phi(\rho) := \int_{\mathbb{R}^d} \rho(x) \log \rho(x)\, dx.$$

It follows from the theory developed in [2] that the curve $(0, \infty) \ni t \mapsto \mu_t \in \mathcal{P}_2(\mathbb{R}^d)$, with density ρ_t given by

$$\rho_t(x) := \int_{\mathbb{R}^d} \frac{1}{(4\pi t)^{d/2}}\, \mathrm{e}^{-|x-y|_2^2/(4t)}\, d\mu_0(y), \quad t > 0,\ x \in \mathbb{R}^d,$$

where $\mu_0 \in \mathcal{P}_2(\mathbb{R}^d)$, satisfies (EVI) with the L_2-Wasserstein metric, the functional (GBE) and $\alpha = 0$.

Surely there are several ways to prove this result. We have chosen for a way which does not make use of the existence theory for (EVI) since we already have a candidate for a solution. Therefore these Lecture Notes are independent of the Lecture Notes (II).

In Section 1 we recall the definition of the Wasserstein metric $W_2(\cdot,\cdot)$ and show that the heat equation induces a C_0-contraction semigroup on $(\mathcal{P}_2(\mathbb{R}^d), W_2(\cdot,\cdot))$. We also give some properties of the metric $W_2(\cdot,\cdot)$ which are useful in the sequel. Finally we show that the curve $t \to \mu_t$ defined above is a (locally) absolutely continuous curve in $(\mathcal{P}_2(\mathbb{R}^d), W_2(\cdot,\cdot))$.

In Section 2, we give a representation formula for the derivative of the (locally) absolutely continuous real-valued function $(0, \infty) \ni t \mapsto \frac{1}{2}(W(\mu_t, \sigma))^2$ where σ is any element of $\mathcal{P}_2(\mathbb{R}^d)$. This approach motivates the study of Theorem 8.4.7 of [2] which treats the case of an arbitrary (locally) absolutely continuous curve in L_p-Wasserstein spaces. For the derivation of this representation formula we need basic results of the theory of Optimal Transport which are used without proofs, but precise references are given.

In Section 3 we first establish conditions which should be satisfied by the functional in (EVI). In particular, we show that under mild assumptions the functional should be of the form given in (GBE). This functional is defined in a precise way in Section 3.3.

In Section 3.1 we also recall a theorem of Daneri–Savaré [12] which shows that the functional should satisfy what is called the “displacement convexity” property. This very interesting notion, introduced by McCann in [17] plays a very important rôle in the study of the “sub-differential” of the functional ϕ and of its approximation in Section 3.2. The final result essentially follows from a combination of the representation formula mentioned above, and of the expression of the “sub-differential” of the functional. It is worth mentioning that the approach used in these notes can be used for other equations than the heat equation. Nevertheless, it would be interesting to know whether there exists a “short” proof of the main result stated in Theorem 3.1.

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1 Heat flow in spaces of probability measures

1.1 Heat flow in the space of probability measures $(\mathcal{P}(\mathbb{R}^d), \beta(\cdot,\cdot))$

Let $\mathbb{R}^d$ be the d-dimensional euclidean space with innerproduct $\langle\cdot,\cdot\rangle$ and corresponding norm $|\cdot|_2$ and let $\mathcal{B}(\mathbb{R}^d)$ denote the collection of all Borel subsets of $\mathbb{R}^d$. Let $\mathcal{P}(\mathbb{R}^d)$ denote the set of all Borel probability measures on $\mathbb{R}^d$ and let $N(0,tI)$, $t \geq 0$, denote the Gaussian probability measures on $\mathbb{R}^d$ with mean 0 and covariance matrix tI. In particular, $N(0,0)$ is δ_0, the Dirac measure at 0, and for $t>0$, $N(0,tI)$ is the measure with density

$$p_t(x) := \frac{1}{(2\pi t)^{d/2}}\,\exp\Big(\frac{-|x|_2^2}{2t}\Big), \quad t>0, \quad x\in\mathbb{R}^d. \tag{1.1}$$

We recall that if $\mu_1,\mu_2\in\mathcal{P}(\mathbb{R}^d)$, the convolution of μ_1 and μ_2, denoted by $\mu_1*\mu_2$, defined by

$$(\mu_1*\mu_2)(B) = \int_{\mathbb{R}^d}\mu_1(B-s)\,\mu_2(ds), \quad B\in\mathcal{B}(\mathbb{R}^d) \tag{1.2}$$

is again an element of $\mathcal{P}(\mathbb{R}^d)$. In particular, $\mu_1*\delta_0=\mu_1$. Moreover, for $\mu_1,\mu_2,\mu_3\in\mathcal{P}(\mathbb{R}^d)$ we have $\mu_1*(\mu_2*\mu_3)=(\mu_1*\mu_2)*\mu_3$ and $\mu_1*\mu_2=\mu_2*\mu_1$.

We also recall that the Gaussian family of measures $\{N(0,tI)\}_{t\geq 0}$ satisfies the "infinite divisibility" property, namely

$$N(0,(t+s)I) = N(0,tI)*N(0,sI), \quad t,s\geq 0.$$

As a consequence, the family of mappings (operators) $S_\beta(t):\mathcal{P}(\mathbb{R}^d)\to\mathcal{P}(\mathbb{R}^d)$ defined by

$$S_\beta(t)\mu_0 := \mu_0 * N(0,\beta t I), \quad t\geq 0, \tag{1.3}$$

where β is a positive number, and $\mu_0\in\mathcal{P}(\mathbb{R}^d)$, satisfies the *semigroup property*

$$\begin{aligned} &S_\beta(t+s) = S_\beta(t)\big(S_\beta(s)\big), \quad t,s\geq 0,\\ &S_\beta(0) = I, \end{aligned} \tag{1.4}$$

where I denotes the identity operator in $\mathcal{P}(\mathbb{R}^d)$.

We shall also use the notation

$$\mu_t := S_\beta(t)\mu_0, \quad t>0. \tag{1.5}$$

Observe that for $t>0$, μ_t is absolutely continuous with respect to the Lebesgue measure λ_d on $\mathbb{R}^d$, and that its density denoted by ρ_t satisfies

$$\rho_t(x) = \int_{\mathbb{R}^d} p_{\beta t}(x-y)\,\mu_0(dy), \quad x\in\mathbb{R}^d. \tag{1.6}$$

The subspace of $\mathcal{P}(\mathbb{R}^d)$ consisting of absolutely continuous measures with respect to λ_d will be denoted by $\mathcal{P}^a(\mathbb{R}^d)$. We use the notation $\mu=\rho\cdot\lambda_d$ for $\mu\in\mathcal{P}^a(\mathbb{R}^d)$ and ρ its density in $L^1(\mathbb{R}^d)$.

As a consequence of (1.6) we have

$$S_\beta(t)\mathcal{P}(\mathbb{R}^d)\subset\mathcal{P}^a(\mathbb{R}^d), \quad t>0. \tag{1.7}$$

Next we recall that the function $(0,\infty)\times\mathbb{R}^d \ni (t,x) \mapsto \rho_t(x) \in \mathbb{R}$ is C^∞, satisfies the *heat equation*

$$\frac{\partial}{\partial t}\rho_t(x) = \frac{\beta}{2}\,\Delta\rho_t(x), \tag{1.8}$$

$$\frac{1}{(2\pi\beta t)^{d/2}} > \rho_t(x) > 0 \quad\text{and}\quad \int_{\mathbb{R}^d}\rho_t(x)\,dx = 1. \tag{1.9}$$

The semigroup $\{S_\beta(t)\}_{t\ge 0}$ will be called the *heat semigroup* on $\mathcal{P}(\mathbb{R}^d)$. We investigate further continuity properties of this semigroup.

Let us first consider the case where the "initial value" $\mu_0 \in \mathcal{P}^a(\mathbb{R}^d)$. Let $\rho_0 \in L^1(\mathbb{R}^d)$ be such that $\mu_0 = \rho_0\cdot\lambda_d$ and let ρ_t be given by (1.6) for $t>0$, then it is known that the map $[0,\infty)\ni t\mapsto \rho_t \in L^1(\mathbb{R}^d)$ is continuous with respect to the L^1-distance. Let

$$D := \Big\{\rho \in L^1(\mathbb{R}^d) : \rho \ge 0 \quad\text{and}\quad \int_{\mathbb{R}^d}\rho\,dx = 1\Big\}$$

and let $J : \mathcal{P}^a(\mathbb{R}^d) \to D$ be defined by $J\mu := \rho$, where ρ is the density of μ. For $\mu^1,\mu^2 \in \mathcal{P}^a(\mathbb{R}^d)$ set

$$d_{\mathrm{TV}}(\mu^1,\mu^2) := \int_{\mathbb{R}^d} |J\mu^1 - J\mu^2|\,dx.$$

Using the fact that J is a bijection and that D is a closed subspace of $L^1(\mathbb{R}^d)$, one verifies that $(\mathcal{P}^a(\mathbb{R}^d), d_{\mathrm{TV}}(\cdot,\cdot))$ is a complete metric space and that for $\mu_0\in\mathcal{P}^a(\mathbb{R}^d)$

$$[0,\infty)\ni t\mapsto S_\beta(t)\mu_0 \in (\mathcal{P}(\mathbb{R}^d), d_{\mathrm{TV}})$$

is continuous. Using Young's inequality for convolutions we have

$$d_{\mathrm{TV}}(S_\beta(t)\mu^1, S_\beta(t)\mu^2) \le d_{\mathrm{TV}}(\mu^1,\mu^2), \quad t>0,$$

where $\mu^1,\mu^2\in\mathcal{P}(\mathbb{R}^d)$.

We introduce the following definition.

Definition 1.1. Let (X,d) be a metric space. A family $\{S(t)\}_{t\ge 0}$ of mappings (operators) from X into X satisfying

(i) the semigroup property

$$S(t)(S(s)x) = S(t+s)x, \quad t,s\ge 0,\ x\in X,$$

(ii) the contraction property

$$d\big(S(t)x^1, S(t)x^2\big) \le d(x^1,x^2), \quad t\ge 0,\ x^1,x^2\in X,$$

(iii) the continuity property

$$[0,\infty)\ni t\mapsto S(t)x\in X \quad\text{is continuous, } x\in X,$$

is called a C_0*-contraction semigroup* on X.

In particular the heat semigroup on $(\mathcal{P}_a(X), d_{\mathrm{TV}}(\cdot,\cdot))$ is a C_0-contraction semigroup.

We now consider the case where $\mu_0 \in \mathcal{P}(\mathbb{R}^d)$.

First we show that the metric $d_{\mathrm{TV}}(\cdot,\cdot)$ is the restriction to $\mathcal{P}^a(\mathbb{R}^d)$ of a metric, still denoted by $d_{\mathrm{TV}}(\cdot,\cdot)$, called "total variation" metric and defined on $\mathcal{P}(\mathbb{R}^d)$. We shall see that the heat semigroup on $\big(\mathcal{P}(\mathbb{R}^d), d_{\mathrm{TV}}(\cdot,\cdot)\big)$ is a contraction semigroup but not a C_0-semigroup. Given $\mu^1, \mu^2 \in \mathcal{P}^a(\mathbb{R}^d)$ with $\mu^i = f^i \cdot \lambda_d$, $i = 1, 2$, we have by definition

$$d_{\mathrm{TV}}(\mu^1, \mu^2) = \int_{\mathbb{R}^d} |f^1 - f^2| \, dx.$$

Notice

$$\int_{\mathbb{R}^d} |f^1 - f^2| \, dx = \sup\Big\{ \Big| \int_{\mathbb{R}^d} (f^1 - f^2) g \, dx \Big| : \; g \in \mathrm{BC}(\mathbb{R}^d), \; \|g\|_\infty \le 1 \Big\},$$

where $\mathrm{BC}(\mathbb{R}^d)$ is the space of bounded continuous real-valued functions on $\mathbb{R}^d$, and $\|\cdot\|_\infty$ is the supremum norm.

This leads to the following definition:

$$d_{\mathrm{TV}}(\mu^1, \mu^2) := \sup\Big\{ \Big| \int_{\mathbb{R}^d} g \, d\mu^1 - \int_{\mathbb{R}^d} g \, d\mu^2 \Big| : g \in \mathrm{BC}(\mathbb{R}^d), \; \|g\|_\infty \le 1 \Big\}. \tag{1.10}$$

It turns out that $(\mathcal{P}(\mathbb{R}^d), d_{\mathrm{TV}}(\cdot,\cdot))$ is a complete metric space and that $\mathcal{P}^a(\mathbb{R}^d)$ is a closed subset. Moreover, $d_{\mathrm{TV}}(\mu^1, \mu^2)$ is the total variation of the signed measure $\mu^1 - \mu^2$ [13]. We also recall that in (1.10) one can replace the space of test functions $\mathrm{BC}(\mathbb{R}^d)$ by the smaller space $\mathrm{BL}(\mathbb{R}^d)$, the space of bounded Lipschitz continuous real-valued functions on $\mathbb{R}^d$. Now we show that

$$d_{\mathrm{TV}}(S_\beta(t)\mu^1, S_\beta(t)\mu^2) \le d_{\mathrm{TV}}(\mu^1, \mu^2) \quad \text{for every } t > 0, \ \mu^1, \mu^2 \in \mathcal{P}(\mathbb{R}^d). \tag{1.11}$$

Let $g \in \mathrm{BC}(\mathbb{R}^d)$ with $\|g\|_\infty \le 1$. Noticing that

$$\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |g(x+y)| p_{\beta t}(x) \, dx \, d\mu^i(y) \le 1, \quad i = 1, 2,$$

we have

$$\begin{aligned}
&\Big| \int_{\mathbb{R}^d} g(x) \big(S_\beta(t)\mu^1\big)(dx) - \int_{\mathbb{R}^d} g(x) \big(S_\beta(t)\mu^2\big)(dx) \Big| \\
&= \Big| \int_{\mathbb{R}^d} g(x) \int_{\mathbb{R}^d} p_{\beta t}(x-y) \, d\mu^1(y) \, dx - \int_{\mathbb{R}^d} g(x) \int_{\mathbb{R}^d} p_{\beta t}(x-y) \, d\mu^2(y) \, dx \Big| \\
&= \Big| \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} g(x+y) p_{\beta t}(x) \, dx \, d\mu^1(y) - \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} g(x+y) p_{\beta t}(x) \, dx \, d\mu^2(y) \Big| \\
&= \Big| \int_{\mathbb{R}^d} \Big(\int_{\mathbb{R}^d} g(x+y) \, d\mu^1(y) - \int_{\mathbb{R}^d} g(x+y) \, d\mu^2(y) \Big) p_{\beta t}(x) \, dx \Big| \\
&\le \int_{\mathbb{R}^d} \Big| \int_{\mathbb{R}^d} g(x+y) \, d\mu^1(y) - \int_{\mathbb{R}^d} g(x+y) \, d\mu^2(y) \Big| p_{\beta t}(x) \, dx \\
&\le \int_{\mathbb{R}^d} d_{\mathrm{TV}}(\mu^1, \mu^2) p_{\beta t}(y) \, dy = d_{\mathrm{TV}}(\mu^1, \mu^2).
\end{aligned}$$

Taking the supremum over g we obtain (1.11).

Next we consider the continuity of $t \mapsto \mu_t$ defined in (1.5). This function is continuous on $(0,\infty)$. Indeed, given $t>0$, $t_n \geq \frac{t}{2}$, $n \geq 1$, with $t_n \to t$, we have

$$d_{\mathrm{TV}}(\mu_{t_n}, \mu_t) = d_{\mathrm{TV}}\big(S_\beta(t_n - \tfrac{t}{2}) S_\beta(\tfrac{t}{2})\mu_0, S_\beta(\tfrac{t}{2}) S_\beta(\tfrac{t}{2})\mu_0\big) \to 0,$$

since $S_\beta(\frac{t}{2})\mu_0 \in \mathcal{P}^a(\mathbb{R}^d)$ and $S_\beta(\cdot)$ is a C_0-semigroup on $\big(\mathcal{P}^a(\mathbb{R}^d), d_{\mathrm{TV}}(\cdot,\cdot)\big)$. Clearly the function $t \mapsto \mu_t$ is continuous at 0 if $\mu_0 \in \mathcal{P}^a(\mathbb{R}^d)$ and cannot be continuous at 0 if $\mu_0 \notin \mathcal{P}^a(\mathbb{R}^d)$ since $\mathcal{P}^a(\mathbb{R}^d)$ is closed in $\big(\mathcal{P}(\mathbb{R}^d), d_{\mathrm{TV}}(\cdot,\cdot)\big)$.

There exists on $\mathcal{P}(\mathbb{R}^d)$ a weaker notion of convergence, the "narrow convergence", for which the function $t \mapsto \mu_t$ is continuous at 0. We recall

Definition 1.2. A sequence $(\mu^n) \subset \mathcal{P}(\mathbb{R}^d)$ is *narrowly convergent* to $\mu \in \mathcal{P}(\mathbb{R}^d)$ as $n \to \infty$ if

$$\lim_{n\to\infty} \int_{\mathbb{R}^d} g(x)\, d\mu^n(x) = \int_{\mathbb{R}^d} g(x)\, d\mu(x) \quad \text{for every } g \in \mathrm{BC}(\mathbb{R}^d). \tag{1.12}$$

We shall use the notation $\mu^n \Rightarrow \mu$ for narrow convergence. We recall (see [2]) that in (1.12) the class $\mathrm{BC}(\mathbb{R}^d)$ can be replaced by the smaller class $C_c^\infty(\mathbb{R}^d)$.

Notice that the narrow convergence only depends on the topology (open sets) of $\mathbb{R}^d$ and not on the specific (euclidean) metric. We recall an equivalent definition.

Let $(\mu^n) \subset \mathcal{P}(\mathbb{R}^d)$ and $\mu \in \mathcal{P}(\mathbb{R}^d)$. Then $\mu^n \Rightarrow \mu$ iff

$$\int_{\mathbb{R}^d} g(x)\, d\mu(x) \leq \varliminf_{n\to\infty} \int_{\mathbb{R}^d} g(x)\, d\mu^n(x) \tag{1.13}$$

for every function $g : \mathbb{R}^d \to [0,\infty]$ such that $\{x \in \mathbb{R}^d : g(x) \leq c\}$ is closed for every $c \in \mathbb{R}$ (lower semicontinuity, l.s.c).

Since for $\mu^1, \mu^2 \in \mathcal{P}(\mathbb{R}^d)$, $\int_{\mathbb{R}^d} g\, d\mu^1 = \int_{\mathbb{R}^d} g\, d\mu^2$ for every $g \in \mathrm{BC}(\mathbb{R}^d)$ (equivalently $g \in C_c^\infty(\mathbb{R}^d)$) implies $\mu^1 = \mu^2$, it follows that $\mu^n \Rightarrow \mu$ and $\mu^n \Rightarrow \widehat{\mu}$ implies $\mu = \widehat{\mu}$.

The narrow convergence can be characterized by metrics on $\mathcal{P}(\mathbb{R}^d)$ ([13]). In these lectures we shall use the so-called β metric ([13]) defined as follows. Let $\mu^1, \mu^2 \in \mathcal{P}(\mathbb{R}^d)$, then

$$\beta(\mu^1, \mu^2) := \sup\Big\{\Big|\int_{\mathbb{R}^d} g\, d\mu^1 - \int_{\mathbb{R}^d} g\, d\mu^2\Big| \;:\; g \in \mathrm{BL}(\mathbb{R}^d), \|g\|_\infty + [g]_{\mathrm{Lip}} \leq 1\Big\}, \tag{1.14}$$

where in (1.14) $[g]_{\mathrm{Lip}}$ is the Lipschitz constant of g. Using what precedes it is easy to verify that $\beta(\cdot,\cdot)$ is a metric on $\mathcal{P}(\mathbb{R}^d)$ and that

$$\beta(\mu^1, \mu^2) \leq d_{\mathrm{TV}}(\mu^1, \mu^2), \quad \mu^1, \mu^2 \in \mathcal{P}(\mathbb{R}^d). \tag{1.15}$$

Moreover, if $(\mu^n) \subset \mathcal{P}(\mathbb{R}^d)$, $\mu \in \mathcal{P}(\mathbb{R}^d)$ and $\lim_{n\to\infty} \beta(\mu^n, \mu) = 0$, then $\mu^n \Rightarrow \mu$. It turns out that the converse is also true, i.e. $\mu^n \Rightarrow \mu$ implies $\beta(\mu^n, \mu) \to 0$ (see [13], [14]). In the next proposition we collect properties about the space $(\mathcal{P}(\mathbb{R}^d), \beta)$ and the heat semigroup on $\mathcal{P}(\mathbb{R}^d)$.

Proposition 1.1. (i) *The space $(\mathcal{P}(\mathbb{R}^d), \beta)$ is a complete metric space.*

(ii) *Convex combinations with rational coefficients of point measures δ_x with $x \in \mathbb{Q}^d$ are dense in $(\mathcal{P}(\mathbb{R}^d), \beta)$. In particular, $(\mathcal{P}(\mathbb{R}^d), \beta)$ is separable.*

(iii) *A subset $\mathcal{C} \subset \mathcal{P}(\mathbb{R}^d)$ is relatively compact with respect to the metric β iff for every $\varepsilon > 0$ there exists K_ε compact subset of $\mathbb{R}^d$ such that for every $\mu \in \mathcal{C}$ we have $\mu(\mathbb{R}^d \setminus K_\varepsilon) \leq \varepsilon$ (tightness).*

(iv) $\mu^n \Rightarrow \mu$ *iff* $\int_{\mathbb{R}^d} g\, d\mu^n \to \int_{\mathbb{R}^d} g\, d\mu$ *for every* $g \in C_c^\infty(\mathbb{R}^d) \iff \beta(\mu^n, \mu) \to 0$.

(v) $\beta(\mu^1, \mu^2) \leq d_{\mathrm{TV}}(\mu^1, \mu^2)\ \forall \mu^1, \mu^2 \in \mathcal{P}(\mathbb{R}^d)$.

(vi) $\beta(\sigma * \mu^1, \sigma * \mu^2) \leq \beta(\mu^1, \mu^2)$ *for every* $\sigma, \mu^1, \mu^2 \in \mathcal{P}(\mathbb{R}^d)$.

(vii) $\lim\limits_{t \to 0} \beta(N(0, tI), \delta_0) = 0$.

(viii) *The heat semigroup on* $(\mathcal{P}(\mathbb{R}^d), \beta)$ *is a* C_0*-contraction semigroup.*

(ix) $\mathcal{P}^a(\mathbb{R}^d)$ *is dense in* $(\mathcal{P}(\mathbb{R}^d), \beta)$.

Proof. (i), (ii), (iii), (iv) see [13, Section 11], [14], [2] for the first equivalence in (iv).

(v) is trivial.

(vi) Let g be as in (1.14). Then

$$\begin{aligned}
&\Big|\int_{\mathbb{R}^d} g\, d(\sigma * \mu^1) - \int_{\mathbb{R}^d} g\, d(\sigma * \mu^2)\Big| \\
&\qquad = \Big|\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} g(x+y)\, d\mu^1(x)\, d\sigma(y) - \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} g(x+y)\, d\mu^2(x)\, d\sigma(y)\Big| \\
&\qquad \leq \int_{\mathbb{R}^d}\Big|\int_{\mathbb{R}^d} g(x+y)\, d\mu^1(x) - \int_{\mathbb{R}^d} g(x+y)\, d\mu^2(x)\Big|\, d\sigma(y) \\
&\qquad \leq \int_{\mathbb{R}^d} \beta(\mu^1, \mu^2)\, d\sigma(y) = \beta(\mu^1, \mu^2).
\end{aligned}$$

Taking the supremum over g we get (vi).

(vii) Let g be as in (1.14) and $\varepsilon > 0$, $t > 0$.

$$\begin{aligned}
\Big|\int_{\mathbb{R}^d} g(x)\rho_t(x)\, dx - g(0)\Big| &= \Big|\int_{\mathbb{R}^d} (g(x) - g(0))\rho_t(x)\, dx\Big| \\
&\leq \int_{B(0,\varepsilon)} |g(x) - g(0)|\rho_t(x)\, dx + \int_{\mathbb{R}^d \setminus B(0,\varepsilon)} |g(x) - g(0)|\rho_t(x)\, dx \\
&\leq \varepsilon + 2\int_{\mathbb{R}^d \setminus B(0,\varepsilon)} \rho_t(x)\, dx.
\end{aligned}$$

Taking the supremum over functions g and $\overline{\lim\limits_{t \to 0}}$ we obtain

$$\overline{\lim_{t \to 0}}\, \beta\big(N(0, tI), \delta_0\big) \leq \varepsilon,$$

which implies (vii).

(viii) (vi) implies the contractivity of $S_\beta(t)$, $t \geq 0$, which together with (vii) implies the continuity of $t \mapsto \mu_t$ on $[0, \infty)$.

(ix) follows from $S_\beta(t)(\mathcal{P}(\mathbb{R}^d)) \subset \mathcal{P}^a(\mathbb{R}^d)$, $t > 0$, and the continuity of $t \mapsto \mu_t$ at $t = 0$. □

1.2 Heat flow in the space of probability measures $(\mathcal{P}_p(\mathbb{R}^d), W_p(\cdot,\cdot))$

In the preceding section we have seen that the heat flow in $\mathcal{P}(\mathbb{R}^d)$ satisfies

$$\int_{\mathbb{R}^d} f(x) S_\beta(t)\,\mu_0(dx) \xrightarrow{t\to 0} \int_{\mathbb{R}^d} f(x)\,\mu_0(dx) \tag{1.16}$$

for every $\mu_0 \in \mathcal{P}(\mathbb{R}^d)$ and every $f \in \mathrm{BC}(\mathbb{R}^d)$. In this section we shall show that the class of functions f for which (1.16) holds can be enlarged provided that the initial measure μ_0 possesses a finite absolute p-moment, i.e. satisfies

$$\exists p > 0 \text{ such that } m_p(\mu_0) := \int_{\mathbb{R}^d} |x|_2^p \, d\mu_0(x) < \infty. \tag{1.17}$$

We shall prove that in that case (1.16) holds for continuous functions $f : \mathbb{R}^d \to \mathbb{R}$ with p-growth, i.e. functions $f : \mathbb{R}^d \to \mathbb{R}$ satisfying

$$\exists C_1, C_2 \geq 0 \text{ such that } |f(x)| \leq C_1 + C_2 |x|_2^p, \quad x \in \mathbb{R}^d. \tag{1.18}$$

It will be shown in Section 1.3 that this convergence can be induced by a metric.

We introduce the notations

$$\begin{aligned} \mathcal{P}_p(\mathbb{R}^d) &:= \{\mu \in \mathcal{P}(\mathbb{R}^d) : m_p(\mu) < \infty\}, \\ \mathcal{P}_p^a(\mathbb{R}^d) &:= \mathcal{P}_p(\mathbb{R}^d) \cap \mathcal{P}^a(\mathbb{R}^d) \quad \text{for } p > 0. \end{aligned} \tag{1.19}$$

We have

$$m_p(\delta_x) = |x|_2^p, \quad x \in \mathbb{R}^d, \quad p > 0, \tag{1.20}$$

and

$$\begin{gathered} m_p(N(0,tI)) = a_{p,d}\, t^{p/2}, \quad t, p > 0 \\ \text{where } a_{p,d} = 2^{p/2-1} \pi^{-d/2}\, d V_d\, \Gamma\Big(\frac{p+d}{2}\Big) \\ \text{and } V_d = \begin{cases} \dfrac{\pi^n}{n!} & \text{if } d = 2n \\ \dfrac{2^n \pi^{n-1}}{1 \cdot 3 \cdot 5 \cdots (2n-1)} & \text{if } d = 2n-1 \end{cases} \end{gathered} \tag{1.21}$$

In particular

$$m_2(N(0,tI)) = dt, \quad t \geq 0.$$

Now we show that $\mathcal{P}_p(\mathbb{R}^d)$ is invariant under the heat semigroup $S_\beta(\cdot)$. More generally we have

Lemma 1.1. *Let $p > 0$ and let $\mu, \sigma \in \mathcal{P}_p(\mathbb{R}^d)$. Then*

$$m_p(\sigma * \mu) \leq 2^p m_p(\sigma) + 2^p m_p(\mu). \tag{1.22}$$

In particular, for $\mu_0 \in \mathcal{P}_p(\mathbb{R}^d)$ and $t > 0$:

$$m_p(S_\beta(t)) \leq 2^p m_p(N(0, \beta t I)) + 2^p m_p(\mu_0) < \infty. \tag{1.23}$$

Proof. Setting $f_{n,p}(x) := |x|_2^p \wedge n$, $n \geq 1$, and noticing that $f_{n,p} \in \mathrm{BC}(\mathbb{R}^d)$, we have for $t > 0$:

$$\begin{aligned}\int_{\mathbb{R}^d} f_{n,p}(x)\, d(\sigma * \mu)(x) &= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f_{n,p}(x+y)\, d\sigma(x)\, d\mu(y) \\ &\leq \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} (2^p|x|_2^p \wedge n + 2^p|y|_2^p \wedge n) d\sigma(x)\, d\mu(y) \leq 2^p m_p(\sigma) + 2^p m_p(\mu).\end{aligned}$$

Then (1.22) follows from the monotone convergence theorem and (1.23) is a consequence of (1.22) and (1.21). □

Next we introduce a metric on $\mathcal{P}_p(\mathbb{R}^d)$, $p \geq 1$, which is stronger than the β metric. Set

$$\mathcal{A} := \big\{g \in \mathrm{BL}(\mathbb{R}^d) : \|g\|_\infty + [g]_{\mathrm{Lip}} \leq 1\big\}.$$

Let $p \geq 1$ and $\mu^1, \mu^2 \in \mathcal{P}_p(\mathbb{R}^d)$. Then

$$\beta(\mu^1, \mu^2) = \sup\Big\{\Big|\int_{\mathbb{R}^d} g\, d\mu^1 - \int_{\mathbb{R}^d} g\, d\mu^2\Big| : g \in \mathcal{A}\Big\}.$$

Notice that

$$\int_{\mathbb{R}^d} g\, d\mu^1 - \int_{\mathbb{R}^d} g\, d\mu^2 = \int_{\mathbb{R}^d\times\mathbb{R}^d} (g(x) - g(y))\, d\gamma(x,y) \tag{1.24}$$

where γ is any Borel probability measure on $\mathbb{R}^d \times \mathbb{R}^d$ ($\gamma \in \mathcal{P}(\mathbb{R}^d \times \mathbb{R}^d)$) satisfying

$$\gamma(A \times \mathbb{R}^d) = \mu^1(A), \quad \gamma(\mathbb{R}^d \times A) = \mu^2(A) \quad \text{for every } A \in \mathcal{B}(\mathbb{R}^d). \tag{1.25}$$

We shall use the notation

$$\Gamma(\mu^1, \mu^2) := \{\gamma \in \mathcal{P}(\mathbb{R}^d \times \mathbb{R}^d) : (1.25) \text{ holds}\}, \quad \text{for } \mu^1, \mu^2 \in \mathcal{P}(\mathbb{R}^d). \tag{1.26}$$

Clearly we have

$$\mu^1 \otimes \mu^2 \in \Gamma(\mu^1, \mu^2). \tag{1.27}$$

One verifies that if μ^1 and/or μ^2 is a Dirac measure then

$$\Gamma(\mu^1, \mu^2) = \{\mu^1 \otimes \mu^2\}. \tag{1.28}$$

We can estimate the left hand side of (1.24) as follows (recalling that $p \geq 1$ and $g \in \mathcal{A}$):

$$\begin{aligned}\Big|\int_{\mathbb{R}^d} g\, d\mu^1 - \int_{\mathbb{R}^d} g\, d\mu^2\Big| &\leq \int_{\mathbb{R}^d\times\mathbb{R}^d} |g(x) - g(y)|\, d\gamma(x,y) \\ &\leq \int_{\mathbb{R}^d\times\mathbb{R}^d} |x-y|_2\, d\gamma(x,y) \leq \Big(\int_{\mathbb{R}^d\times\mathbb{R}^d} |x-y|_2^p\, d\gamma(x,y)\Big)^{1/p} \\ &\leq \Big(2^{p-1}\int_{\mathbb{R}^d} (|x|^p + |y|^p)\, d\gamma(x,y)\Big)^{1/p} = 2^{1-1/p}\Big(m_p(\mu^1) + m_p(\mu^2)\Big)^{1/p} < \infty.\end{aligned}$$

Taking the supremum over $g \in \mathcal{A}$ we obtain

$$\beta(\mu^1, \mu^2) \leq \inf\Big\{\Big(\int_{\mathbb{R}^d\times\mathbb{R}^d} |x-y|_2^p\, d\gamma(x,y)\Big)^{1/p} : \gamma \in \Gamma(\mu^1, \mu^2)\Big\}. \tag{1.29}$$

Definition 1.3. Let $\mu^1, \mu^2 \in \mathcal{P}_p(\mathbb{R}^d)$ with $p \geq 1$. Then

$$W_p(\mu^1, \mu^2) := \inf\Big\{ \Big(\int_{\mathbb{R}^d \times \mathbb{R}^d} |x - y|_2^p \, d\gamma(x, y) \Big)^{1/p} : \gamma \in \Gamma(\mu^1, \mu^2) \Big\}$$

where $\Gamma(\mu^1, \mu^2)$ is as in (1.26).

The function $W_p(\cdot, \cdot) : \mathcal{P}_p(\mathbb{R}^d) \times \mathcal{P}_p(\mathbb{R}^d) \to [0, \infty)$ is clearly symmetric and satisfies: $W_p(\mu^1, \mu^2) = 0$ implies $\mu^1 = \mu^2$. Indeed, since β is a metric and $W_p(\mu^1, \mu^2) \geq \beta(\mu^1, \mu^2)$ we get $\mu^1 = \mu^2$. Moreover, we have $W_p(\mu, \mu) = 0$, since the probability measure $\widetilde{\gamma} \in \mathcal{P}(\mathbb{R}^d \times \mathbb{R}^d)$ defined by

$$\widetilde{\gamma}(B) := \mu\big(\{x \in \mathbb{R}^d : (x, x) \in B\}\big)$$

satisfies $\widetilde{\gamma} \in \Gamma(\mu, \mu)$ and $\int_{\mathbb{R}^d \times \mathbb{R}^d} |x - y|^p \, d\widetilde{\gamma}(x, y) = 0$. Less trivial is the fact that the function $W_p(\cdot, \cdot)$ satisfies the triangle inequality (see e.g. [2], [13], or for an elementary proof [9]), and therefore $W_p(\cdot, \cdot)$ is a metric on $\mathcal{P}_p(\mathbb{R}^d)$. Notice that if $1 \leq p_1 < p_2$ we have $\mathcal{P}_{p_2}(\mathbb{R}^d) \subset \mathcal{P}_{p_1}(\mathbb{R}^d)$ and $W_{p_1}(\mu^1, \mu^2) \leq W_{p_2}(\mu^1, \mu^2)$, $\mu^1, \mu^2 \in \mathcal{P}_{p_2}(\mathbb{R}^d)$.

Moreover if, in the definition of the metric β, (1.14), we weaken the condition $\|g\|_\infty + [g]_{\mathrm{Lip}} \leq 1$ by requiring only $[g]_{\mathrm{Lip}} \leq 1$, we obtain for $\mu^1, \mu^2 \in \mathcal{P}_1(\mathbb{R}^d)$:

$$\sup\Big\{ \Big| \int_{\mathbb{R}^d} g \, d\mu^1 - \int_{\mathbb{R}^d} g \, d\mu^2 \Big| : g \in \mathrm{BL}(\mathbb{R}^d),\ [g]_{\mathrm{Lip}} \leq 1 \Big\} \leq W_1(\mu^1, \mu^2) < \infty. \tag{1.30}$$

A celebrated theorem of Kantorovich and Rubinstein [13, 11.8] implies that equality holds in (1.30). The metrics $W_p(\cdot, \cdot)$ are usually called Kantorovich–Rubinstein–Wasserstein metrics. For a discussion on this terminology, see the footnote of [2, page 1].

Remark. Replacing $(\mathbb{R}^d, |\cdot|_2)$ by any separable metric space (X, d) and defining

$$\mathcal{P}_p(X) := \Big\{ \mu \in \mathcal{P}(X) : \int_X (d(x, x_0))^p \, d\mu(x) < \infty \text{ for some (equivalently all) } x_0 \in X \Big\}$$

one can prove that $W_p(\cdot, \cdot)$ defined as above (replacing $|x - y|_2^p$ by $(d(x, y))^p$, $p \geq 1$) is a metric on $\mathcal{P}_p(X)$. The Kantorovich-Rubinstein theorem holds in this generality. For $p \in (0, 1)$, $W_p^p(\cdot, \cdot)$ is a metric on $\mathcal{P}_p(X)$.

Now we consider the type of convergence induced by the metric $W_p(\cdot, \cdot)$ on $\mathcal{P}_p(\mathbb{R}^d)$, $p \geq 1$. Let $\mu^n, \mu \in \mathcal{P}_p(\mathbb{R}^d)$ be such that $\lim_{n \to \infty} W_p(\mu^n, \mu) = 0$. In view of (1.29) and Proposition 1.1(iv) we have $\mu^n \Rightarrow \mu$.

Moreover, since

$$\big| W_p(\delta_0, \mu^n) - W_p(\delta_0, \mu) \big| \leq W_p(\mu^n, \mu) \to 0$$

and $W_p(\delta_0, \mu^n) = m_p^{1/p}(\mu^n)$ (similarly for μ), we have

$$\lim_{n \to \infty} m_p(\mu^n) = m_p(\mu). \tag{1.31}$$

It will be shown in the next section that if $\mu^n \Rightarrow \mu$ and (1.31) holds then $W_p(\mu^n, \mu) \to 0$. Even more $W_p(\mu^n, \mu) \to 0$ is equivalent to (1.12) for functions g satisfying (1.18).

Next we consider the analogue of Proposition 1.1(vi) for the metric $W_p(\cdot, \cdot)$.

Lemma 1.2 (see Lemma 4.9 of [10])**.** *Let $p \geq 1$ and $\sigma, \mu^1, \mu^2 \in \mathcal{P}_p(\mathbb{R}^d)$. Then we have*

$$W_p(\sigma * \mu^1, \sigma * \mu^2) \leq W_p(\mu^1, \mu^2). \tag{1.32}$$

As a consequence the heat semigroup $S_\beta(\cdot)$ is a contraction semigroup on $(\mathcal{P}_p(\mathbb{R}^d), W_p(\cdot, \cdot))$.

Proof. By Lemma 1.1 $\sigma * \mu^i \in \mathcal{P}_p(\mathbb{R}^d)$, $i = 1,2$. Let $\varepsilon > 0$ and X_1, X_2 be $\mathbb{R}^d$-valued random vectors whose laws are μ^1, μ^2, respectively, and $\mathbb{E}(|X_1 - X_2|^p) \leq W_p^p(\mu^1, \mu^2) + \varepsilon$. Let Y be a $\mathbb{R}^d$-valued random vector independent of X_1 and X_2 such that σ is the law of Y. Then $\sigma * \mu^i$ is the law of $X_i + Y$, $i = 1,2$. Consequently,

$$W_p^p(\sigma * \mu^1, \sigma * \mu^2) \leq \mathbb{E}\big(|(X_1+Y) - (X_2+Y)|^p\big) = \mathbb{E}\big(|X_1 - X_2|^p\big) \leq W_p^p(\mu^1, \mu^2) + \varepsilon. \ \square$$

Another consequence of (1.32) is the right-continuity of $S_\beta(t)$ at $t = 0$. Indeed, for $t > 0$:

$$\begin{aligned} W_p(S_\beta(t)\mu_0, \mu_0) &= W_p(N(0,\beta t I) * \mu_0, \delta_0 * \mu_0) \\ &\leq W_p(N(0,\beta t I), \delta_0) = m_p^{1/p}(N(0,\beta t I)) = (a_{p,d})^{1/p}(\beta t)^{1/2} \end{aligned}$$

by (1.21).

It is easy to verify that a contraction semigroup which is right-continuous at 0 is a C_0-semigroup, therefore the heat semigroup $S_\beta(\cdot)$ is a C_0-semigroup on $(\mathcal{P}_p(\mathbb{R}^d), W_p(\cdot,\cdot))$, $p \geq 1$.

We investigate further continuity properties of $S_\beta(\cdot)$. Let $0 \leq s < t$. Clearly

$$W_p(S_\beta(t)\mu_0, S_\beta(s)\mu_0) \leq W_p(S_\beta(t-s)\mu_0, \mu_0) \leq (a_{p,d})^{1/p}\beta^{1/2}(t-s)^{1/2},$$

which implies $\frac{1}{2}$-Hölder continuity of $S_\beta(\cdot)$. A better estimate holds. Indeed by using (1.32) again we obtain

$$W_p(S_\beta(t)\mu_0, S_\beta(s)\mu_0) \leq W_p(N(0,\beta t I), N(0,\beta s I)).$$

Arguing as in Lemma 4.6 of [10] we consider a $\mathbb{R}^d$-valued gaussian random vector X with law $N(0,I)$. Then

$$W_p^p(N(0,\beta t I), N(0,\beta s I)) \leq \mathbb{E}\big(|(\beta t)^{1/2}X - (\beta s)^{1/2}X|^p\big) = \big|(\beta t)^{1/2} - (\beta s)^{1/2}\big|^p \mathbb{E}(|X|^p).$$

Hence $W_p(N(0,\beta t I), N(0,\beta s I)) \leq |t^{1/2} - s^{1/2}|\beta^{1/2}a_{p,d}^{1/p}$ (see Appendix A2).

Therefore we have

$$W_p(S_\beta(t)\mu_0, S_\beta(s)\mu_0) \leq \beta^{1/2}a_{p,d}^{1/p}\int_s^t \frac{1}{2} r^{-1/2}\, dr \quad \text{for } 0 \leq s < t. \tag{1.33}$$

It is proved in [15] that equality holds when $p = 2$.

We recall the following

Definition 1.4. Let (X,d) be a metric space and $a, b \in \mathbb{R}$ with $a < b$. A function $u : [a,b] \to X$ is called *absolutely continuous* on $[a,b]$ if to each $\varepsilon > 0$ there corresponds a $\delta > 0$ such that, for all positive integers n and all families $(a_1,b_1), \ldots, (a_n,b_n)$ of disjoint open subintervals of $[a,b]$ of total length at most δ, we have

$$\sum_{k=1}^n d(u(a_k), u(b_k)) \leq \varepsilon. \tag{1.34}$$

The collection of all such functions is denoted by $\mathrm{AC}([a,b];X)$.

Observe that $\mathrm{AC}([a,b];X) \subset C([a,b];X)$.

We recall a fundamental result of real analysis.

Theorem 1.1.

i) *Let* $u \in \mathrm{AC}([a,b];\mathbb{R})$. *Then* u *is differentiable a.e. in* (a,b), $u' \in L^1(a,b)$ *and*

$$\int_s^t u'(r)\,dr = u(t) - u(s) \quad \text{for all } a \le s < t \le b. \tag{1.35}$$

ii) *Let* $f \in L^1(a,b)$. *Then the function* $t \mapsto u(t) = \int_a^t f(r)\,dr$ *is absolutely continuous on* $[a,b]$ *and* $u'(t) = f(t)$ *a.e. in* (a,b).

Remark. The following generalization of Theorem 1.1 holds. Let X be a reflexive Banach space (in particular a Hilbert space).

(i) If $u \in \mathrm{AC}([a,b];X)$ then u is strongly differentiable a.e. in (a,b), $u' \in L^1(a,b;X)$ and (1.35) holds where the integral is a Bochner integral.

(ii) If $f \in L^1(a,b;X)$, $u(t) := \int_a^t f(s)\,ds$, $t \in [a,b]$, then $u \in \mathrm{AC}([a,b];X)$ and $u'(t) = f(t)$ a.e. in (a,b).

The following characterization of absolute continuity will be very useful.

Theorem 1.2 (see Appendix A1). *Let* $u : [a,b] \to X$, (X,d) *a metric space. Then* $u \in \mathrm{AC}([a,b];X)$ *iff there exists* $m \in L^1(a,b)$, $m \ge 0$, *such that*

$$d(u(s),u(t)) \le \int_s^t m(r)\,dr \quad \text{for all } a \le s < t \le b. \tag{1.36}$$

Moreover, if $u \in \mathrm{AC}([a,b];X)$,

$$|\dot{u}|(t) := \lim_{h\to 0} \frac{d(u(t+h),u(t))}{|h|}$$

exists for almost all $t \in (a,b)$, $|\dot{u}| \in L^1(a,b)$,

$$d(u(s),u(t)) \le \int_s^t |\dot{u}|(r)\,dr, \quad a \le s \le t \le b,$$

and if m *satisfies* (1.36), *then* $|\dot{u}|(r) \le m(r)$ *a.e.*

The function $t \mapsto |\dot{u}|(t) \in \mathbb{R}_+$ is called the *metric derivative* of u.

Remark. If $u \in \mathrm{AC}([a,b];X)$ then $u \in \mathrm{BV}([a,b];X)$ (bounded variation) and $\int_a^b |\dot{u}|(t)\,dt = \mathrm{Var}(u;[a,b])$ the (total) variation of u on $[a,b]$.

If $u \in \mathrm{AC}([a,b];X)$ then it is easy to verify that the function $t \mapsto v(t) := d(u(t),z)$, $z \in X$, belongs to $\mathrm{AC}([a,b];\mathbb{R})$, as well as $t \mapsto (v(t))^2$.

As a consequence, for every $\sigma \in \mathcal{P}_2(\mathbb{R}^d)$, $T > 0$, the function

$$[0,T] \ni t \mapsto \tfrac{1}{2} W_2^2(S_\beta(t)\mu_0, \sigma)$$

is absolutely continuous and the metric derivative of $\mu_t := S_\beta(t)\mu_0$ exists a.e. in $(0,T)$.

1.3 Further properties of the space $(\mathcal{P}_p(\mathbb{R}^d), W_p(\cdot,\cdot))$

We introduce some notations and definitions.

Let (X,d) be a separable metric space with its Borel σ-algebra $\mathcal{B}(X)$ and let $\mathcal{P}(X)$ be the set of Borel probability measures on X. If (Y,d') is another separable metric space, if $\mu \in \mathcal{P}(X)$ and $f : X \to Y$ is Borel measurable, then $f_\#\mu$ denotes the image measure on $\mathcal{B}(Y)$, i.e. $f_\#\mu(A) := \mu(f^{-1}(A))$ for $A \in \mathcal{B}(Y)$. We recall that if $g : Y \to [0,\infty)$ is Borel measurable, we have

$$\int_X g(f(x))\, d\mu(x) = \int_Y g(y)\, d(f_\#\mu)(y).$$

In particular if $X = Y = \mathbb{R}^d$ and $f(x) = 0$ for all $x \in \mathbb{R}^d$, then for any $\mu \in \mathcal{P}(\mathbb{R}^d)$ we have

$$\int_{\mathbb{R}^d} g(y)\, d(f_\#\mu)(y) = \int_{\mathbb{R}^d} g(f(x))\, d\mu(x) = \int_{\mathbb{R}^d} g(0)\, d\mu(x) = g(0)$$

for every $g \in \mathrm{BC}(\mathbb{R}^d)$. Therefore $f_\#\mu = \delta_0$ the Dirac measure at 0. So if $\mu \in \mathcal{P}^a(\mathbb{R}^d)$, $f_\#\mu$ does not need to belong to $\mathcal{P}^a(\mathbb{R}^d)$. For a sufficient condition for $f_\#\mu$ to belong to $\mathcal{P}^a(\mathbb{R}^d)$, see e.g. Appendix 2.

Considering the product $X_1 \times X_2$ of two separable metric spaces (in particular we shall identify $\mathbb{R}^d \times \mathbb{R}^d$ with $\mathbb{R}^{2d}$ equipped with the euclidean metric), we define the canonical projections $\pi^i : X_1 \times X_2 \to X_i$, by

$$\pi^i(x_1, x_2) = x_i, \quad i = 1, 2.$$

Similarly, for product of three spaces, $\pi^{i,j} : X_1 \times X_2 \times X_3 \to X_i \times X_j$,

$$\pi^{i,j}(x_1, x_2, x_3) = (x_i, x_j), \quad 1 \le i < j \le 3.$$

If $\gamma \in \mathcal{P}(X_1 \times X_2)$, then

$$\pi^1_\#\gamma(A) = \gamma(A \times X_2), \quad \pi^2_\#\gamma(B) = \gamma(X_1 \times B), \quad A \in \mathcal{B}(X_1),\ B \in \mathcal{B}(X_2).$$

Given $\mu^i \in \mathcal{P}(X_i)$, $i = 1, 2$, we define

$$\Gamma(\mu^1, \mu^2) := \{\gamma \in \mathcal{P}(X_1 \times X_2) : \pi^i_\#\gamma = \mu^i,\ i = 1, 2\}$$

which is consistent with (1.26).

Now we return to the case $X_1 = X_2 = \mathbb{R}^d$ and we can define the β metric on $\mathcal{P}(\mathbb{R}^d \times \mathbb{R}^d)$ as the β metric on $\mathcal{P}(\mathbb{R}^{2d})$. Observe that the maps $\gamma \mapsto \pi^i_\#\gamma$ are continuous from $\big(\mathcal{P}(\mathbb{R}^d \times \mathbb{R}^d), \beta\big) \to \big(\mathcal{P}(\mathbb{R}^d), \beta\big)$. Indeed, if $\gamma, \gamma^1, \ldots, \gamma^n \in \mathcal{P}(\mathbb{R}^d \times \mathbb{R}^d)$ and $\gamma^n \Rightarrow \gamma$, then for any $g : \mathbb{R}^d \to \mathbb{R}$ continuous and bounded we have

$$\int_{\mathbb{R}^d} g(x)\, d(\pi^1_\#\gamma^n)(x) = \int_{\mathbb{R}^d\times\mathbb{R}^d} g(\pi^1 y)\, d\gamma^n(y) \to \int_{\mathbb{R}^d\times\mathbb{R}^d} g(\pi^1 y)\, d\gamma(y) = \int_{\mathbb{R}^d} g(x)\, d(\pi^1_\#\gamma)(x).$$

Similarly for $\pi^2_\#$. As a consequence the set $\Gamma(\mu^1, \mu^2)$ is closed in $(\mathcal{P}(\mathbb{R}^d \times \mathbb{R}^d), \beta)$. We shall show that it is even compact. In view of Proposition 1.1(iii) it is sufficient to show that it is tight. By the same Proposition 1.1(iii), but with the reverse implication, we have: for every $\varepsilon > 0$ there exist compact subsets of $\mathbb{R}^d$, $K_{1,\varepsilon}$ and $K_{2,\varepsilon}$ such that $\mu^1(\mathbb{R}^d \setminus K_{1,\varepsilon}) \le \varepsilon/2$, $\mu^2(\mathbb{R}^d \setminus K_{2,\varepsilon}) \le \varepsilon/2$. Set $K_\varepsilon := K_{1,\varepsilon} \times K_{2,\varepsilon}$ which is compact in $\mathbb{R}^d \times \mathbb{R}^d$. Note that

$$\mathbb{R}^d \times \mathbb{R}^d \setminus K_\varepsilon \subset (\mathbb{R}^d \setminus K_{1,\varepsilon}) \times \mathbb{R}^d \cup \mathbb{R}^d \times (\mathbb{R}^d \setminus K_{2,\varepsilon}).$$

Hence for any $\gamma \in \Gamma(\mu^1, \mu^2)$ we have

$$\begin{aligned}\gamma(\mathbb{R}^d \times \mathbb{R}^d \setminus K_\varepsilon) \le \gamma\big((\mathbb{R}^d \setminus K_{1,\varepsilon}) \times \mathbb{R}^d\big) + \gamma\big(\mathbb{R}^d \times (\mathbb{R}^d \setminus K_{2,\varepsilon})\big) \\ = \mu^1(\mathbb{R}^d \setminus K_{1,\varepsilon}) + \mu^2(\mathbb{R}^d \setminus K_{2,\varepsilon}) \le \varepsilon.\end{aligned}$$

This implies the tightness, hence the compactness of $\Gamma(\mu^1, \mu^2)$.

Finally, noticing that the map

$$\Gamma(\mu^1, \mu^2) \ni \gamma \mapsto \int_{\mathbb{R}^d \times \mathbb{R}^d} |x_1 - x_2|_2^p \, d\gamma(x^1, x^2) \in \mathbb{R}, \quad p \ge 1,$$

is l.s.c., by (1.13), it has a global minimizer. Therefore in Definition 1.3 we can replace the right hand side of (1.29) by

$$\min\Big\{ \Big(\int_{\mathbb{R}^d \times \mathbb{R}^d} |x - y|_2^p \, d\gamma(x, y) \Big)^{1/p} : \gamma \in \Gamma(\mu^1, \mu^2) \Big\}.$$

Now we consider a slightly more general situation where we have two narrowly convergent sequences in $\mathcal{P}(\mathbb{R}^d)$, $\mu^n \Rightarrow \mu$, $\sigma^n \Rightarrow \sigma$, and we are interested in the union of the sets $\Gamma(\mu^n, \sigma^n)$. Since the sets $\{\mu, \mu^1, \dots, \mu^n, \dots\}$ and $\{\sigma, \sigma^1, \dots, \sigma^n, \dots\}$ are compact in $(\mathcal{P}(\mathbb{R}^d), \beta)$, we obtain as above the tightness of $\bigcup_{n \ge 1} \Gamma(\mu^n, \sigma^n)$. Therefore there exist $\gamma \in \mathcal{P}(\mathbb{R}^d \times \mathbb{R}^d)$ and a subsequence $n(k)$ such that $\gamma_{n(k)} \in \Gamma(\mu^{n(k)}, \sigma^{n(k)})$ and $\gamma_{n(k)} \Rightarrow \gamma$. Moreover, in view of the continuity of $\pi^i_\#$ we get $\gamma \in \Gamma(\mu, \sigma)$. We shall use this fact in order to prove the completeness of the space $\big(\mathcal{P}_p(\mathbb{R}^d), W_p(\cdot, \cdot)\big)$, $p \ge 1$. Indeed, let μ_n be a Cauchy sequence in $\big(\mathcal{P}_p(\mathbb{R}^d), W_p(\cdot, \cdot)\big)$. By (1.29) it is also a Cauchy sequence in $(\mathcal{P}(\mathbb{R}^d), \beta)$ and by Proposition 1.1(i) it has a β-limit $\mu \in \mathcal{P}(\mathbb{R}^d)$.

Given $\varepsilon > 0$, let $N \ge 1$ be such that $W_p(\mu_m, \mu_n) \le \varepsilon$ for all $m, n \ge N$. We claim that given $\overline{x} \in \mathbb{R}^d$, $n \ge N$, we have

$$\int_{\mathbb{R}^d} |x - \overline{x}|_2^p \, d\mu(x) \le 2^p \varepsilon + 2^p \int_{\mathbb{R}^d} |y - \overline{x}|_2^p \, d\mu_n(y), \tag{1.37}$$

$$W_p(\mu, \mu_n) \le \varepsilon, \tag{1.38}$$

which implies $\mu \in \mathcal{P}_p(\mathbb{R}^d)$ and $W_p(\mu_n, \mu) \to 0$. Let $m, n \ge 1$. By what precedes there exists $\gamma_{m,n} \in \Gamma(\mu_m, \mu_n)$ such that

$$W_p^p(\mu_m, \mu_n) = \int_{\mathbb{R}^d \times \mathbb{R}^d} |x - y|_2^p \, d\gamma_{m,n}(x, y). \tag{1.39}$$

Now let n be fixed. Applying our previous result with the two sequences μ_m and $\sigma_m := \mu_n$ for every $m \ge 1$, with $\mu_m \Rightarrow \mu$ and $\sigma_m \Rightarrow \mu_n$, we find $\gamma_n \in \Gamma(\mu, \mu_n)$ and a subsequence $\gamma_{m_k,n} \in \Gamma(\mu_{m_k}, \mu_n)$ satisfying $\gamma_{m_k,n} \Rightarrow \gamma_n$.

Let $\overline{x} \in \mathbb{R}^d$. Then, for $m_k, n \ge N$

$$\begin{aligned}\int_{\mathbb{R}^d} |x - \overline{x}|_2^p \, d\mu_{m_k}(x) = \int_{\mathbb{R}^d \times \mathbb{R}^d} |x - \overline{x}|_2^p \, d\gamma_{m_k,n} \\ \le 2^p \int_{\mathbb{R}^d \times \mathbb{R}^d} |x - y|_2^p \, d\gamma_{m_k,n} + 2^p \int_{\mathbb{R}^d \times \mathbb{R}^d} |y - \overline{x}|_2^p \, d\gamma_{m_k,n} \le 2^p \varepsilon^p + 2^p \int_X |y - \overline{x}|_2^p \, \delta\mu_n.\end{aligned}$$

Since $\mu_{m_k} \Rightarrow \mu$, claim (1.37) follows from (1.13). Similarly,

$$\int_{\mathbb{R}^d\times\mathbb{R}^d} |x-y|_2^p \, d\gamma_n(x,y) \le \varliminf_{k\to\infty} \int_{\mathbb{R}^d\times\mathbb{R}^d} |x-y|_2^p \, d\gamma_{m_k,n}(x,y) = \varliminf_{k\to\infty} W_p^p(\mu_{m_k}, \mu_n) \le \varepsilon^p.$$

Since $\gamma_n \in \Gamma(\mu, \mu_n)$, we have

$$W_p^p(\mu, \mu_n) \le \int_{\mathbb{R}^d\times\mathbb{R}^d} |x-y|_2^p \, d\gamma_n(x,y) \le \varepsilon^p,$$

which proves (1.38).

In what follows, as in [2], we shall use the notation

$$\Gamma_0(\mu^1, \mu^2) := \Big\{ \gamma \in \Gamma(\mu^1,\mu^2) : \int_{\mathbb{R}^d\times\mathbb{R}^d} |x-y|_2^p \, d\gamma(x,y) = W_p^p(\mu^1,\mu^2) \Big\}, \tag{1.40}$$

where $\mu^1, \mu^2 \in \mathcal{P}_p(\mathbb{R}^d)$. As above we consider two sequences $\mu^n \Rightarrow \mu$, $\sigma^n \Rightarrow \sigma$ and the corresponding sets $\Gamma(\mu^n, \sigma^n)$. Let $\gamma^n \in \Gamma_0(\mu^n, \sigma^n)$, $n \ge 1$, and let a subsequence $\gamma^{n(k)}$, $\gamma \in \Gamma(\mu, \sigma)$ be such that $\gamma^{n(k)} \Rightarrow \gamma$. A natural question is whether $\gamma \in \Gamma_0(\mu, \sigma)$. The answer is affirmative and follows from a characterization of the global minimizers of the functional

$$\gamma \mapsto \int_{\mathbb{R}^d\times\mathbb{R}^d} |x-y|^p \, d\gamma(x,y), \quad \gamma \in \Gamma(\mu^1,\mu^2).$$

We state this characterization without giving a proof. For a proof see e.g. [2] or [14].

We need the following definition

Definition 1.5. Let (X,d) be a separable metric space and let $\mu \in \mathcal{P}(X)$. We define the *support* of μ by

$$\operatorname{supp}\mu := \{x \in X : \mu(B_r(x) > 0) \text{ for every } r > 0\}. \tag{1.41}$$

One verifies that $\operatorname{supp}\mu$ is closed and $\mu(\operatorname{supp}\mu) = 1$.

We recall

Theorem 1.3. *Let $p \ge 1$, $\mu^1, \mu^2 \in \mathcal{P}_p(\mathbb{R}^d)$ and let $\gamma \in \Gamma(\mu^1,\mu^2)$. Then $\gamma \in \Gamma_0(\mu^1,\mu^2)$ (see (1.40)) iff $\operatorname{supp}\gamma$ satisfies: for every $n = 1, 2, \ldots$*

$$\sum_{i=1}^n |x_{\sigma(i)} - y_i|_2^p \ge \sum_{i=1}^n |x_i - y_i|_2^p \tag{1.42}$$

for every $(x_i, y_i) \in \operatorname{supp}\gamma$, $i = 1, 2, \ldots, n$, and for every permutation σ of $\{1, \ldots, n\}$.

Returning to the question mentioned above we have by one implication in Theorem 1.1 that $\operatorname{supp}\gamma^{n(x)}$ satisfies (1.42) for every $k \ge 1$.

As a consequence of the other implication in Theorem 1.1 we need to show that $\operatorname{supp}\gamma$ satisfies (1.42). This follows from

Lemma 1.3 ([2, Proposition 5.1.8, p. 112])**.** *Let (X,d) be a separable metric space and let $\mu, \mu_1, \ldots, \mu_n, \ldots \in \mathcal{P}(X)$ be such that $\mu_n \Rightarrow \mu$. Then for every $x \in \operatorname{supp}\mu$, there exists a sequence $\{x_n\} \subset X$ such that $x_n \in \operatorname{supp}\mu_n$, $n \ge 1$, and $\lim_{n\to\infty} x_n = x$.*

Proof. Let $x \in \operatorname{supp} \mu$. Then for every $k \geq 1$ $\mu(B_{1/k}(x)) > 0$. Since the characteristic function of the open ball $B_{1/k}(x)$ is l.s.c., we have by (1.13)

$$\varliminf_{m \to \infty} \mu_m(B_{1/k}(x)) \geq \mu(B_{1/k}(x)).$$

Hence for every $k \geq 1$ there exists $m_k(x)$ such that $\mu_m(B_{1/k}(x)) > 0$ for every $m \geq m_k(x)$. Set $n_0(x) := m_1(x)$ and

$$n_k(x) := \max(n_{k-1}(x) + 1, m_{k+1}(x)) \quad \text{for } k \geq 1.$$

We have $n_k(x) < n_{k+1}(x)$, $k \geq 0$, and

$$\mu_n(B_{1/k}(x)) > 0 \quad \text{for } n \geq n_{k-1}(x),\ k \geq 1.$$

It follows that for $k \geq 1$, $n \geq n_{k-1}(x)$, $\operatorname{supp} \mu_n \cap B_{1/k}(x) \neq \varnothing$. Choose $x_n \in \operatorname{supp} \mu_n \cap B_{1/k}(x)$ for $n_{k-1}(x) \leq n < n_k(x)$. Choose $x_n \in \operatorname{supp} \mu_n$ arbitrary for $1 \leq n < n_0(x)$. Clearly $\lim x_n = x$. □

We summarize these results in the next proposition.

Proposition 1.2. *Let $p \geq 1$.*

(i) *Let $\mu, \sigma \in \mathcal{P}_p(\mathbb{R}^d)$. Then $\Gamma_0(\mu, \sigma)$ defined in* (1.40) *is not empty.*

(ii) *Let $\mu^n, \mu, \sigma^n, \sigma \in \mathcal{P}_p(\mathbb{R}^d)$, $n \geq 1$, be such that $\mu^n \Rightarrow \mu$ and $\sigma^n \Rightarrow \sigma$.*

Let $\gamma^n \in \Gamma_0(\mu^n, \sigma^n)$, $n \geq 1$. Then there exists $\gamma \in \Gamma_0(\mu, \sigma)$ and a subsequence $\gamma^{n(k)}$ such that $\gamma^{n(k)} \Rightarrow \gamma$ in $\mathcal{P}(\mathbb{R}^d \times \mathbb{R}^d)$.

In particular $\Gamma_0(\mu, \sigma)$ is compact in $(\mathcal{P}(\mathbb{R}^d \times \mathbb{R}^d), \beta)$.

We conclude this section by giving an analogue of Proposition 1.1 for the space $(\mathcal{P}_p(\mathbb{R}^d), W_p(\cdot, \cdot))$.

Proposition 1.3. *Let $p \geq 1$.*

(i) *The space $(\mathcal{P}_p(\mathbb{R}^d), W_p(\cdot, \cdot))$ is a complete metric space.*

(ii) $W_p(\delta_x, \delta_y) = |x - y|_2$, $x, y \in \mathbb{R}^d$.

(iii) *Let $\mu, \mu^n \in \mathcal{P}_p(\mathbb{R}^d)$, $n \geq 1$. Then $W_p(\mu^n, \mu) \to 0$ iff $\mu^n \Rightarrow \mu$ and $m_p(\mu^n) \to m_p(\mu)$ iff $\int_{\mathbb{R}^d} f \, d\mu^n \to \int_{\mathbb{R}^d} f \, d\mu$ for every $f \in C(\mathbb{R}^d)$ satisfying* (1.18).

(iv) *Convex combinations with rational coefficients of point measures δ_x with $x \in \mathbb{Q}^d$ are dense in $(\mathcal{P}_p(\mathbb{R}^d), W_p(\cdot, \cdot))$.*

(v) $W_p(\mu^1, \mu^2) \leq \beta(\mu^1, \mu^2)$, $\mu^1, \mu^2 \in \mathcal{P}_p(\mathbb{R}^d)$.

(vi) $W_p(\sigma * \mu^1, \sigma * \mu^2) \leq W_p(\mu^1, \mu^2)$, $\sigma, \mu^1, \mu^2 \in \mathcal{P}_p(\mathbb{R}^d)$.

(vii) $\lim_{t \to 0} W_p(N(0, tI), \delta_0) = 0$.

(viii) *The heat semigroup on $(\mathcal{P}_p(\mathbb{R}^d), W_p(\cdot, \cdot))$ is a C_0-contraction semigroup.*

(ix) *$\mathcal{P}^a(\mathbb{R}^d) \cap \mathcal{P}_p(\mathbb{R}^d)$ is dense in $(\mathcal{P}_p(\mathbb{R}^d), W_p(\cdot, \cdot))$.*

(x) *For every $T > 0$, the function*

$$[0,T] \ni t \mapsto \mu_t := S_\beta(t)\mu_0 \in (\mathcal{P}_p(\mathbb{R}^d), W_p(\cdot,\cdot)),$$

with $\mu_0 \in \mathcal{P}_p(\mathbb{R}^d)$, is absolutely continuous.

Proof. (i) has been proved above.

(v)–(x) have been proved in Section 1.2.

(ii) follows from $\Gamma(\delta_x, \delta_y) = \{\delta_x \otimes \delta_y\}$.

It remains to prove (iii) and (iv).

(iii) The implication $W_p(\mu^n, \mu) \to 0 \Rightarrow (\mu^n \Rightarrow \mu$ and $(m_p(\mu^n) \to m_p(\mu))$ has been proved in Section 1.2. Now we prove:

$$\mu^n \Rightarrow \mu \text{ and } m_p(\mu^n) \to m_p(\mu), \quad \mu, \mu^n \in \mathcal{P}_p(\mathbb{R}^d),\ n \geq 1, \tag{1.43}$$

implies

$$\int_{\mathbb{R}^d} f\, d\mu^n \to \int_{\mathbb{R}^d} f\, d\mu \quad \text{for every } f \in C(\mathbb{R}^d) \text{ satisfying (1.18).} \tag{1.44}$$

Assuming (1.43) we first establish

$$\lim_{k\to\infty} \sup_{n\geq 1} \int_{\{x\in\mathbb{R}^d : |x|_2^p \geq k\}} |x|_2^p \, d\mu_n(x) = 0, \tag{1.45}$$

(i.e. in the terminology of [2], the set $\{\mu^n\} \subset \mathcal{P}(\mathbb{R}^d)$ has uniformly integrable p-moments).

Set $F^k := \{x \in \mathbb{R}^d : |x|_2^p \geq k\}$, $k \geq 1$. Notice that since F^k is closed, its characteristic function is upper semicontinuous and (1.13) yields

$$\overline{\lim_{n\to\infty}}\, \mu^n(F^k) \leq \mu(F^k).$$

Moreover, setting $f_k(x) := |x|_2^p \wedge k$, $k \geq 1$, and $f(x) := |x|_2^p$, we have

$$\int_{F^k} f\, d\mu^n = \int_{\mathbb{R}^d} (f - f_k)\, d\mu^n + k\mu^n(F^k),$$

similarly for μ instead of μ^n. Since $f_k \in \mathrm{BC}(\mathbb{R}^d)$, (1.12) yields

$$\lim_{n\to\infty} \int_{\mathbb{R}^d} f_k\, d\mu^n = \int_{\mathbb{R}^d} f_k\, d\mu.$$

Therefore for every $k \geq 1$, we have

$$\begin{aligned}\overline{\lim_{n\to\infty}} \int_{F^k} f\, d\mu^n &= \overline{\lim_{n\to\infty}} \Big[\int_{\mathbb{R}^d} (f - f_k)\, d\mu^n + k\mu^n(F_k)\Big] \\ &\leq \int_{\mathbb{R}^d} (f - f_k)\, d\mu + k\mu(F^k) = \int_{F^k} f\, d\mu = \int_{F^k} |x|_2^p\, d\mu.\end{aligned}$$

Consequently, given $\varepsilon > 0$ there exists $m = m_\varepsilon \geq 1$ such that

$$\overline{\lim_{n\to\infty}} \int_{F^m} |x|_2^p\, d\mu^n(x) \leq \int_{F^m} |x|_2^p\, d\mu \leq \frac{\varepsilon}{2}.$$

Hence there exists $N = N_\varepsilon \geq 1$ such that

$$\int_{F^m} |x|_2^p \, d\mu^n(x) \leq \varepsilon \quad \text{for every } n \geq N.$$

Choosing $l \geq m$ large enough we find

$$\int_{F^l} |x|_2^p \, d\mu^n(x) \leq \varepsilon \quad \text{for } 1 \leq n \leq l.$$

Finally, noticing that $k \mapsto \int_{F^k} |x|_2^p \, d\mu^n(x)$ is nonincreasing for all $n \geq 1$, we obtain

$$\sup_{n\geq 1} \int_{F^l} |x|_2^p \, d\mu^n(x) \leq \varepsilon,$$

as well as

$$\sup_{n\geq 1} \int_{F^k} |x|_2^p \, d\mu^n(x) \leq \varepsilon \quad \text{for all } k \geq l.$$

This implies (1.45).

Next we show that (1.44) holds. This is a consequence of the following

Lemma 1.4. *Let* $\mu, \mu^n \in \mathcal{P}_p(\mathbb{R}^d)$, $n \geq 1$ *for some* $p \geq 1$, *be such that* $\mu^n \Rightarrow \mu$ *and* (1.45) *holds. Then* (1.44) *holds.*

Proof of Lemma 1.4. Taking positive and negative parts of f in (1.44) it is sufficient to consider the case f nonnegative. By (1.13) we obtain

$$\int_{\mathbb{R}^d} f \, d\mu \leq \varliminf_{n\to\infty} \int_{\mathbb{R}^d} f \, d\mu^n.$$

It remains to show

$$\varlimsup_{n\to\infty} \int_{\mathbb{R}^d} f \, d\mu^n \leq \int_{\mathbb{R}^d} f \, d\mu. \tag{1.46}$$

Setting $f_k(x) := f(x) \wedge k$, and $F^k := \{x \in \mathbb{R}^d : f(x) \geq k\}$, we get

$$\int_{\mathbb{R}^d} f \, d\mu^n = \int_{F^k} f \, d\mu^n + \int_{\mathbb{R}^d \setminus F^k} f \, d\mu^n \leq C_1 \mu^n(F^k) + C_2 \int_{F^k} |x|_2^p \, d\mu^n(x) + \int_{\mathbb{R}^d} f_k \, d\mu^n.$$

Using the closedness of F^k and the boundedness of f_k, we get for $k \geq 1$,

$$\begin{aligned} \varlimsup_{n\to\infty} \int_{\mathbb{R}^d} f \, d\mu^n &\leq C_1 \mu(F^k) + C_2 \sup_{n\geq 1} \int_{F^k} |x|_2^p \, d\mu^n(x) + \int_{\mathbb{R}^d} f_k \, d\mu \\ &\leq C_1 \mu(F^k) + C_2 \sup_{n\geq 1} \int_{F^k} |x|_2^p \, d\mu^n(x) + \int_{\mathbb{R}^d} f \, d\mu. \end{aligned}$$

Taking $\varlimsup\limits_{k\to\infty}$, using (1.45) and $\lim\limits_{k\to\infty} \mu(F^k) = 0$, we arrive at (1.46). □

Finally we show that if $\mu, \mu^n \in \mathcal{P}_p(\mathbb{R}^d)$, $n \geq 1$, and (1.44) holds then $W_p(\mu^n, \mu) \to 0$. Choosing $f \in \mathrm{BC}(\mathbb{R}^d)$ in (1.44) we see that $\mu^n \Rightarrow \mu$. We apply Proposition 1.2(ii) with the same μ, μ^n and with $\sigma = \sigma^n = \mu$ for all $n \geq 1$. Let $\gamma^{n(k)} \in \Gamma_0(\mu^n, \mu)$ and $\gamma \in \Gamma_0(\mu, \mu)$ be as in the conclusion of Proposition 1.2(ii). We have $\gamma^{n(k)} \Rightarrow \gamma$ in $\mathcal{P}(\mathbb{R}^d \times \mathbb{R}^d)$.

Choosing $f(x) := |x|_2^p$, $x \in \mathbb{R}^d$, in (1.44) we have (1.43). From the proof of the implication (1.43)⇒(1.44) we obtain (1.45). Now we want to apply Lemma 1.4 to $\gamma, \gamma^{n(k)} \in \mathcal{P}_p(\mathbb{R}^{2d})$. We have to show

$$\lim_{k\to\infty} \sup_{l\geq 1} \int_{\{(x,y)\in\mathbb{R}^d\times\mathbb{R}^d:(|x|_2^2+|y|_2^2)^{p/2}\geq k\}} (|x|_2^2+|y|_2^2)^{p/2}\, d\gamma^{n(l)}(x,y) = 0. \tag{1.47}$$

This is a consequence of the following

Lemma 1.5. *Let $\mu^{1,m}, \mu^{2,m} \in \mathcal{P}_p(\mathbb{R}^d)$, $m \geq 1$ for some $p \geq 1$, be such that*

$$\lim_{k\to\infty} \sup_{m\geq 1} \int_{|x|_2^p\geq k} |x|_2^p\, d\mu^{i,m}(x) = 0, \quad i = 1, 2. \tag{1.48}$$

Let $\gamma^m \in \Gamma(\mu^{1,m}, \mu^{2,m})$, $m \geq 1$. Then

$$\lim_{k\to\infty} \sup_{m\geq 1} \int_{\{(x,y)\in\mathbb{R}^d\times\mathbb{R}^d:(|x|_2^2+|y|_2^2)^{p/2}\geq k\}} (|x|_2^2+|y|_2^2)^{p/2}\, d\gamma^{m}(x,y) = 0. \tag{1.49}$$

Proof of Lemma 1.5. Let $k \geq 1$. Set

$$F^k := \{(x,y) \in \mathbb{R}^d \times \mathbb{R}^d : (|x|_2^2 + |y|_2^2)^{p/2} \geq k\}.$$

Notice that

$$F^k \subset A^k \cup B^k$$

where

$$A^k := \{(x,y) \in \mathbb{R}^d \times \mathbb{R}^d : |x|_2 \geq \tfrac{1}{\sqrt{2}} k^{1/p}\},$$
$$B^k := \{(x,y) \in \mathbb{R}^d \times \mathbb{R}^d : |y|_2 \geq \tfrac{1}{\sqrt{2}} k^{1/p}\}.$$

Moreover, for $x, y \in \mathbb{R}^d$ we have

$$(|x|_2^2 + |y|_2^2)^{p/2} \leq (|x|_2 + |y|_2)^p \leq 2^{p-1}(|x|_2^p + |y|_2^p).$$

Then

$$\int_{F^k} (|x|_2^2 + |y|_2^2)^{p/2}\, d\gamma^m(x,y) \leq \sum_{j=1}^4 I_j$$

where

$$I_1 := \int_{A^k} |x|_2^p\, d\gamma^m(x,y), \quad I_2 := \int_{B^k} |x|_2^p\, d\gamma^m(x,y),$$
$$I_3 := \int_{A^k} |y|_2^p\, d\gamma^m(x,y), \quad I_4 := \int_{B^k} |y|_2^p\, d\gamma^m(x,y).$$

We have

$$I_1 = \int_{|x|_2\geq \frac{1}{\sqrt{2}}k^{1/p}} |x|_2^p\, d\mu^{1,m}(x) \quad \text{and} \quad I_4 = \int_{|y|_2\geq \frac{1}{\sqrt{2}}k^{1/p}} |y|_2^p\, d\mu^{2,m}(y).$$

Then (1.48) yields

$$\lim_{k\to\infty} \sup_{m\geq 1} I_i(k,m) = 0 \quad \text{for } i = 1 \text{ and } i = 4.$$

Let $l \geq 1$. Then

$$I_3(k,m) = \int_{A^k \cap \{|y|_2 \leq l\}} |y|_2^p \, \delta\gamma^m(x,y) + \int_{A^k \cap \{|y|_2 > l\}} |y|_2^p \, \delta\gamma^m(x,y)$$

$$\leq l^p \int_{A^k} d\gamma^m(x,y) + \int_{\{|y|_2 > l\}} |y|_2^p \, d\gamma^m(x,y).$$

Let $k \geq 2^{p/2}$, then

$$\int_{A^k} d\gamma^m(x,y) = \int_{\{|x|_2 \geq \frac{1}{\sqrt{2}} k^{1/p}\}} d\mu^{1,m}(x) \leq \int_{\{|x|_2 \geq \frac{1}{\sqrt{2}} k^{1/p}\}} |x|_2^p \, d\mu^{1,m}(x).$$

Clearly

$$\int_{\{|y|_2 > k\}} |y|_2^p \, d\gamma^m(x,y) = \int_{\{|y|_2 > l\}} |y|_2^p \, d\mu^{2,m}(y).$$

Then by (1.48) we get

$$\varlimsup_{k \to \infty} \sup_{m \geq 1} I_3(k,m) \leq \int_{\{|y|_2 > l\}} |y|_2^p \, d\mu^{2,m}(y).$$

Since $l \geq 1$ is arbitrary we obtain

$$\lim_{k \to \infty} \sup_{m \geq 1} I_3(k,m) = 0.$$

Similarly for I_2. This completes the proof of (1.49). □

As a consequence of Lemma 1.4 we obtain

$$\lim_{k \to \infty} \int_{\mathbb{R}^d \times \mathbb{R}^d} f(x,y) \, d\gamma^{n(k)}(x,y) = \int_{\mathbb{R}^d \times \mathbb{R}^d} f(x,y) \, d\gamma(x,y)$$

for every $f : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ continuous with p-growth, in particular for $f(x,y) := |x-y|_2^p$. Therefore, using the definitions of $\gamma^{n(k)}$ and γ, we obtain

$$\lim_{k \to \infty} W_p^p(\mu^{n(k)}, \mu) = \lim_{k \to \infty} \int_{\mathbb{R}^d \times \mathbb{R}^d} |x-y|_2^p \, d\gamma^{n(k)}(x,y)$$

$$= \int_{\mathbb{R}^d \times \mathbb{R}^d} |x-y|_2^p \, d\gamma(x,y) = W_p^p(\mu,\mu) = 0.$$

Since every subsequence of μ^n has a convergent subsequence converging to the same limit μ, we get

$$\lim_{n \to \infty} W_p(\mu^n, \mu) = 0.$$

This completes the proof of (iii).

(iv) Let $\mu \in \mathcal{P}_p(\mathbb{R}^d)$ with $p \geq 1$ and let $\varepsilon > 0$. Then there exists $R > 0$ such that

$$\int_{\mathbb{R}^d \setminus \overline{B_R(0)}} |x|_2^p \, d\mu(x) \leq \varepsilon^p.$$

Since $\overline{B_R(0)}$ is compact and $\mathbb{Q}^d$ is dense in $\mathbb{R}^d$, there exists a finite collection of (open) balls $B_\varepsilon(x_k)$, $1 \le k \le N$, with $x_k \in \mathbb{Q}^d$ such that

$$\overline{B_R(0)} \subseteq \bigcup_{k=1}^{N} B_\varepsilon(x_k).$$

Set $S_1 := B_\varepsilon(x_1)$ and define $S_k := B_\varepsilon(x_k) \setminus \bigcup_{j<k} B_\varepsilon(x_j)$, $k > 1$. Then the Borel sets S_k are disjoint, possibly empty. By skipping the indices where $S_k = \varnothing$ and renumbering, we obtain $1 \le M \le N$ disjoint, nonempty Borel sets S_k such that

$$\bigcup_{k=1}^{M} S_k \supseteq \overline{B_R(0)}.$$

Define $f : \mathbb{R}^d \to \mathbb{R}^d$ by setting

$$\begin{aligned} f(x) &= 0 \quad \text{whenever } |x|_2 > R, \\ f(x) &= x_k \quad \text{whenever } |x|_2 \le R \text{ and } x \in S_k \text{ for some } k. \end{aligned}$$

Then f is Borel measurable and for $x \in \overline{B_R(0)}$ we have $|x - f(x)|_2 \le \varepsilon$. Hence

$$\int_{\mathbb{R}^d} |x - f(x)|_2^p \, d\mu(x) \le \varepsilon^p \int_{\overline{B_R(0)}} d\mu(x) + \int_{\mathbb{R}^d \setminus \overline{B_R(0)}} |x|_2^p \, d\mu(x) \le 2\varepsilon^p.$$

Notice that $f_\#\mu$ is a convex combination of Dirac measures δ_{x_j} where $x_j \in \mathbb{Q}^d$. In particular $f_\#\mu \in \mathcal{P}_p(\mathbb{R}^d)$. Moreover, the measure $(\mathrm{id} \times f)_\#\mu \in \Gamma(\mu, f_\#\mu)$, hence

$$W_p^p(\mu, f_\#\mu) \le \int_{\mathbb{R}^d \times \mathbb{R}^d} |x - y|_2^p \, d(\mathrm{id} \times f)_\#\mu(x, y) = \int_{\mathbb{R}^d} |x - f(x)|_2^p \, d\mu(x) \le 2\varepsilon^p.$$

Finally we can approximate $f_\#\mu$ by convex combinations of δ_{x_j} with rational coefficients. Indeed, if

$$f_\#\mu = \sum_{j=1}^{M} \alpha_j \delta_{x_j} \quad \text{with } \alpha_j > 0, \ \sum \alpha_j = 1,$$

and $0 < \alpha_j^n \to \alpha_j$ as $n \to \infty$, $1 \le j < M$, where $\alpha_j^n \in \mathbb{Q}$, we can define

$$\mu^n = \frac{1}{\sum_{j=1}^{M} \alpha_j^n} \sum_{j=1}^{M} \alpha_j^n \delta_{x_j}, \quad n \ge 1.$$

Then $\mu^n \in \mathcal{P}_p(\mathbb{R}^d)$, $n \ge 1$, and

$$\lim_{n\to\infty} \int_{\mathbb{R}^d} g \, d\mu^n = \lim_{n\to\infty} \frac{1}{\sum_{j=1}^{M} \alpha_j^n} \sum_{j=1}^{M} \alpha_j^n g(x_j) = \frac{1}{\sum_{j=1}^{M} \alpha_j} \sum_{j=1}^{M} \alpha_j g(x_j) = \int_{\mathbb{R}^d} g(x) \, d(f_\#\mu)(x)$$

for every $g \in C(\mathbb{R}^d)$ with p-growth.

As a consequence of (iii) we have $\lim_{n\to\infty} W_p(\mu^n, f_\#\mu) = 0$. This completes the proof of (iv) as well as the proof of Proposition 1.3. □

2 A representation formula for $\frac{1}{2}\frac{d}{dt}W_2^2(\mu_t,\sigma)$

Recall that the aim of these notes is to show that the curve $(0,\infty) \ni t \mapsto \mu_t \in \mathcal{P}(\mathbb{R}^d)$, defined in (1.5), satisfies (EVI) in $(\mathcal{P}_2(\mathbb{R}^d), W_2(\cdot,\cdot))$, with ϕ defined in (GBE) and $\alpha = 0$ whenever $\mu_0 \in \mathcal{P}(\mathbb{R}^d)$.

In Section 2.1 we introduce the notion of "optimal transport map" which will play an essential rôle in these notes. In Section 2.2 we use the Benamou–Brenier formula to find an estimate of the metric derivative $|\dot{\mu}|(t)$ in terms of the L^2-norm of the "tangent vector" of the curve $t \mapsto \mu_t$ at t. Finally, in Section 2.3 we obtain the representation of $\frac{1}{2}\frac{d}{dt}W_2^2(\mu_t,\sigma)$ given in [2, 8.4.12].

2.1 The "optimal transport map"

Let $\mu, \sigma \in \mathcal{P}_2(\mathbb{R}^d)$. In Section 1.3 we have shown that there exists $\gamma \in \Gamma(\mu,\sigma)$ such that

$$\int_{\mathbb{R}^d\times\mathbb{R}^d} |x-y|_2^2\, d\gamma(x,y) = W_2^2(\mu,\sigma). \tag{2.1}$$

In (1.40) we defined $\Gamma_0(\mu,\sigma)$ to be the collection of all measures $\gamma \in \Gamma(\mu,\sigma)$ satisfying (2.1). An element of $\Gamma_0(\mu,\sigma)$ is usually called an optimal plan. We shall need the following notations and definitions.

Definition 2.1. For $\mu \in \mathcal{P}_2(\mathbb{R}^d)$ we define

$$\begin{aligned} L^2(\mu;\mathbb{R}^d) := \{&\text{equivalence classes } \mu \text{ a.e. of Borel maps } r : \mathbb{R}^d \to \mathbb{R}^d \\ &\qquad\qquad \text{satisfying } \int_{\mathbb{R}^d} |r(x)|_2^2\, d\mu(x) < \infty\} \end{aligned}$$

equipped with the usual norm and innerproduct.

$\operatorname{Tan}_\mu \mathcal{P}_2(\mathbb{R}^d)$ is the closure in $L^2(\mu;\mathbb{R}^d)$ of the linear subspace $\{\nabla\varphi : \varphi \in C_c^\infty(\mathbb{R}^d;\mathbb{R})\}$, where $\nabla\varphi(x) = \big(\partial_{x_1}\varphi(x),\ldots,\partial_{x_d}\varphi(x)\big)^t$, $x \in \mathbb{R}^d$.

Recall that $\mu_t \in \mathcal{P}_2^a(\mathbb{R}^d)$ for $t > 0$.

When $\mu \in \mathcal{P}_2^a(\mathbb{R}^d)$ it turns out that $\Gamma(\mu,\sigma)$ is a singleton which we denote by $\{\gamma_\mu^\sigma\}$ and that the measure γ_μ^σ has the special form

$$\gamma_\mu^\sigma = (\mathrm{id}, r)_\# \mu$$

for some unique element $r \in \operatorname{Tan}_\mu(\mathcal{P}_2(\mathbb{R}^d))$. We shall denote this element by r_μ^σ and it will be called the *optimal transport map* between μ and σ. The following theorem will be essential.

Theorem 2.1 ([2, 6.2.3 and Proposition 8.5.2])**.** *Let $\mu \in \mathcal{P}_2^a(\mathbb{R}^d)$ and $\sigma \in \mathcal{P}_2(\mathbb{R}^d)$. Then*

(i) *there exists a unique $\gamma \in \Gamma(\mu,\sigma)$ satisfying* (2.1)*. It will be denoted by γ_μ^σ;*

(ii) *there exists a unique element $r \in L^2(\mu;\mathbb{R}^d)$ satisfying $\gamma_\mu^\sigma = (\mathrm{id}, r)_\#\mu$. It will be denoted by r_μ^σ. Moreover, $r_\mu^\sigma \in \operatorname{Tan}_\mu(\mathcal{P}_2(\mathbb{R}^d))$;*

(iii) *if the support of σ (see* (1.41)*) is bounded, then there exists $\varphi : \mathbb{R}^d \to \mathbb{R}$ convex, (globally) Lipschitz continuous such that*

$$r_\mu^\sigma = \nabla\varphi \ \mu\text{-}a.e. \tag{2.2}$$

For the proof we refer the reader to [2, 6.2.3 and the proof of Proposition 8.5.2]. See also [14] for (i), (iii) and the first part of (ii).

Now we return to the curve μ_t defined in (1.5) where $\mu_0 \in \mathcal{P}_2(\mathbb{R}^d)$. We already know from Proposition 1.3 that $\mu_t \in \mathcal{P}_2^a(\mathbb{R}^d)$ for $t > 0$ and that for every $T > 0$, $[0, T] \ni t \mapsto \mu_t \in (\mathcal{P}_2(\mathbb{R}^d), W_2(\cdot,\cdot))$ is absolutely continuous. In particular, in view of Theorem 1.2, $|\dot\mu|(t)$ exists for almost all $t \in (0, \infty)$.

Recall that for $t > 0$, $\mu_t \in \mathcal{P}_2^a(\mathbb{R}^d)$ with $\mu_t = \rho_t \cdot \lambda_d$ where ρ_t is defined in (1.6).

From (1.8) we deduce that for $t > 0$, $x \in \mathbb{R}^d$

$$\frac{\partial}{\partial t}\rho_t(x) + \operatorname{div}(\rho_t(x)\boldsymbol{v}_t(x)) = 0 \tag{2.3}$$

where

$$\boldsymbol{v}_t(x) := -\frac{\beta}{2}\,\frac{\nabla\rho_t(x)}{\rho_t(x)} \tag{2.4}$$

using the fact that $\rho_t(x) > 0$.

Clearly $(t, x) \mapsto \boldsymbol{v}_t(x) \in C^\infty((0,\infty)\times\mathbb{R}^d;\mathbb{R}^d)$.

Equation (2.4) is usually called the *continuity equation* and $\boldsymbol{v}_t(\cdot)$ is the velocity field.

Notice that in the case $\mu_0 = \delta_0$, $\boldsymbol{v}_t(x) = \frac{1}{2t}x$, hence $v_t \in L^2(\mu_t;\mathbb{R}^d)$. This holds for any $\mu_0 \in \mathcal{P}_2(\mathbb{R}^d)$. Indeed, let $1 \le k \le d$ and observe that (with $D_k := \frac{\partial}{\partial x_k}$):

$$\begin{aligned} D_k p_{\beta t}(x) &= -(\beta t)^{-1} x_k p_{\beta t}(x),\\ \frac{D_k\rho_t(x)}{\rho_t(x)} &= -\frac{1}{\beta t}\,\frac{\int_{\mathbb{R}^d}(x_k - y_k)p_{\beta t}(x-y)\,d\mu_0(y)}{\int_{\mathbb{R}^d} p_{\beta t}(x-y)\,d\mu_0(y)}\,,\end{aligned}$$

and by Jensen's inequality

$$\left|\frac{D_k\rho_t(x)}{\rho_t(x)}\right|^2 \le \frac{1}{(\beta t)^2}\,\frac{\int_{\mathbb{R}^d}(x_k - y_k)^2 p_{\beta t}(x-y)\,d\mu_0(y)}{\int_{\mathbb{R}^d} p_{\beta t}(x-y)\,d\mu_0(y)}\,.$$

Summing over k we get

$$\left|\frac{\nabla\rho_t(x)}{\rho_t(x)}\right|_2^2 \le \frac{1}{(\beta t)^2}\,\frac{\int_{\mathbb{R}^d}|x-y|_2^2 p_{\beta t}(x-y)\,d\mu_0(y)}{\rho_t(x)}\,,$$

hence, using $|x-y|_2^2 \le 2|x|_2^2 + 2|y|_2^2$,

$$\left|\frac{\nabla\rho_t(x)}{\rho_t(x)}\right|_2^2 \le \frac{2}{(\beta t)^2}\left(|x|_2^2 + g_t(x)\right),$$

where $g_t(x) > 0$, C^∞, satisfies

$$\begin{aligned}\int_{\mathbb{R}^d} g_t(x)\rho_t(x)\,dx &= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d}|y|_2^2 p_{\beta t}(x-y)\,d\mu_0(y)\,dx\\ &= \int_{\mathbb{R}^d}\underbrace{\int_{\mathbb{R}^d} p_{\beta t}(x-y)\,dx}_{=1}\,|y|_2^2\,d\mu_0(y) = m_2(\mu_0) < \infty.\end{aligned}$$

It follows that

$$\int_{\mathbb{R}^d} |v_t(x)|_2^2 \, d\mu_t(x) \le \frac{1}{2t^2} \left[m_2(\mu_t) + m_2(\mu_0)\right] < \infty.$$

One verifies that $(0,\infty) \ni t \mapsto \int_{\mathbb{R}^d} |v_t(x)|^2 \, d\mu_t(x)$ is continuous. We have

$$\int_{T_1}^{T_2} \int_{\mathbb{R}^d} |v_t(x)|_2^2 \, d\mu_t(x) \, dt < \infty, \quad 0 < T_1 < T_2. \tag{2.5}$$

Now we denote by r_h the optimal transport map from μ_t to μ_{t+h} for $t > 0$, i.e.

$$r_h := r_{\mu_t}^{\mu_{t+h}}, \quad t > 0, \quad -t < h. \tag{2.6}$$

Our aim is to prove that for $t > 0$ such that $|\dot{\mu}|(t)$ exists, we have

$$\lim_{h\to 0} \int_{\mathbb{R}^d} \left| \frac{1}{h}(r_h(x) - x) - v_t(x) \right|_2^2 d\mu_t(x) = 0, \tag{2.7}$$

i.e., $s_h \to v_t$ in $L^2(\mu_t; \mathbb{R}^d)$ as $h \to 0$ where

$$s_h(x) := \frac{1}{h}(r_h(x) - x), \quad x \in \mathbb{R}^d, \quad h \neq 0. \tag{2.8}$$

In the remaining part of this section we shall prove that s_h converges *weakly* to v_t in $L^2(\mu_t; \mathbb{R}^d)$ as $h \to 0$.

Since

$$\int_{\mathbb{R}^d} \frac{|x - r_h(x)|_2^2}{h^2} \, d\mu_t = \frac{W_2^2(\mu_t, \mu_{t+h})}{h^2}$$

tends to $|\dot{\mu}|^2(t)$ as $\to 0$, there exists $C > 0$ such that for $0 < |h| \le 1$:

$$\|s_h\|_{L^2(\mu_t;\mathbb{R}^d)} = \|\frac{1}{h}(r_h - \mathrm{id})\|_{L^2(\mu_t;\mathbb{R}^d)} \le C.$$

First we prove that s_h converges weakly as $h \to 0$ to Πv_t, the orthogonal projection in $L^2(\mu_t; \mathbb{R}^d)$ of μ_t on $\mathrm{Tan}_{\mu_t}(\mathcal{P}(\mathbb{R}^2))$ (see Definition 2.1). It is sufficient to show that any converging sequence s_{h_n} tends weakly to Πv_t where $h_n \to 0$ as $n \to \infty$.

Notice that $s_h \in \mathrm{Tan}_{\mu_t}(\mathcal{P}(\mathbb{R}^d))$ since both id ($= r_{\mu_t}^{\mu_t}$) and r_h belong to this subspace. Therefore the weak limit of s_{h_n}, which we shall denote by s_0, belongs to $\mathrm{Tan}_{\mu_t}(\mathcal{P}(\mathbb{R}^2))$. From the continuity equation (2.4) we obtain for every $\varphi \in C_c^\infty(\mathbb{R}^d)$ and $t > 0$:

$$\begin{aligned}\int_{\mathbb{R}^d} \langle \nabla\varphi, v_t \rangle \, d\mu_t &= \int_{\mathbb{R}^d} \langle \nabla\varphi, v_t \rangle \, \rho_t \, dx = -\int_{\mathbb{R}^d} \varphi \operatorname{div}(\rho_t v_t) \, dx \\ &= \int_{\mathbb{R}^d} \varphi \partial_t \rho_t \, dx = \lim_{h\to 0} \frac{1}{h}[\mu_{t+h}(\varphi) - \mu_t(\varphi)].\end{aligned}$$

(Here $\mu_t(\varphi) := \int_{\mathbb{R}^d} \varphi(x) \, d\mu_t(x)$)

On the other hand

$$\begin{aligned}\frac{1}{h}\big(\mu_{t+h}(\varphi) - \mu_t(\varphi)\big) &= \frac{1}{h} \int_{\mathbb{R}^d} [\varphi(r_h(x)) - \varphi(x)] \, d\mu_t(x) \\ &= \frac{1}{h} \int_{\mathbb{R}^d} [\varphi(x + h s_h(x)) - \varphi(x)] \, d\mu_t(x) = \int_{\mathbb{R}^d} \big[\langle \nabla\varphi(x), s_h(x) \rangle + \omega_h(x)\big] \, d\mu_t(x)\end{aligned}$$

with $\omega_h(x)$ Borel measurable satisfying

$$|\omega_h(x)| \le \frac{1}{2}\,|h| d^2 \max_{1\le i,j\le d} \left\| \frac{\partial^2}{\partial x_i \partial x_j}\varphi \right\|_\infty |s_h(x)|_2^2.$$

Notice $\lim\limits_{h\to 0} \int_{\mathbb{R}^d} |\omega_h(x)|\, d\mu_t(x) = 0$.

Therefore we obtain

$$\int_{\mathbb{R}^d} \langle \nabla\varphi, s_0 - v_t\rangle \; d\mu_t = 0$$

for every $\varphi \in C_c^\infty(\mathbb{R}^d)$. Since

$$\int_{\mathbb{R}^d} \langle \nabla\varphi, v_t\rangle \; d\mu_t = \int_{\mathbb{R}^d} \langle \nabla\varphi, \Pi v_t\rangle \; d\mu_t$$

and $s_0 \in \operatorname{Tan}_{\mu_t}(\mathcal{P}_2(\mathbb{R}^d))$, we obtain $s_0 = \Pi v_t$ and

$$\text{weak-}\lim_{h\to 0} \frac{1}{h}(r_h - \mathrm{id}) = \Pi v_t \quad \text{in } L^2(\mu_t;\mathbb{R}^d). \tag{2.9}$$

Next we show that $\Pi v_t = v_t$, i.e.

$$v_t \in \operatorname{Tan}_{\mu_t} \mathcal{P}_2(\mathbb{R}^d) \tag{2.10}$$

(hence $s_h \to v_t$ weakly in $L^2(\mu_t;\mathbb{R}^d)$).

This is a consequence of the following

Lemma 2.1. *Let $\mu \in \mathcal{P}_2^a(\mathbb{R}^d)$ with $\mu = \rho\cdot\lambda_d$. Let $\varphi : \mathbb{R}^d \to \mathbb{R}$ be locally Lipschitz continuous. If $\varphi \in L^1(\mu)$ and $\nabla\varphi \in L^2(\mu;\mathbb{R}^d)$, then $\nabla\varphi \in \operatorname{Tan}_\mu(\mathcal{P}_2(\mathbb{R}^d))$.*

Indeed, let $\mu := \mu_t$, $\rho := \rho_t$ and $\varphi(x) := \log\rho_t(x)$, $x\in\mathbb{R}^d$, in Lemma 2.1. Then

$$0 < \rho_t(x) \le \frac{1}{(2\pi t)^{d/2}} \quad \text{for } t>0 \text{ and } x\in\mathbb{R}^d.$$

Moreover, $\varphi \in C^1(\mathbb{R}^d)$, hence locally Lipschitz continuous. Since $\mu_t \in \mathcal{P}_2(\mathbb{R}^d)$, we have

$$\int_{\{\rho_t(x)<1\}} |\log\rho_t(x)|\,\rho_t(x)\,dx < \infty$$

by Lemma 3.2(ii) and

$$\int_{\{\rho_t(x)\ge 1\}} \log\rho_t(x)\rho_t(x)\,dx < \infty$$

since ρ_t is bounded above. Therefore we have

$$x \mapsto v_t(x) = \frac{-\beta}{2}\nabla\varphi(x) = -\frac{\beta}{2}\,\frac{\nabla\rho_t(x)}{\rho_t(x)} \in \operatorname{Tan}_{\mu_t}(\mathcal{P}_2(\mathbb{R}^d)).$$

Proof of Lemma 2.1.

(i) We truncate the function φ. Let $m\ge 1$ and let $\varphi_m(x) := (\varphi(x)\wedge m)\vee(-m)$, $x\in\mathbb{R}^d$.

Clearly φ_m is also locally Lipschitz continuous and *bounded*, hence $\varphi_m \in L^1(\mu)$ and

$$\nabla\varphi_m(x) = \begin{cases} \nabla\varphi_m(x) & \text{if } |\varphi(x)| \le m, \\ 0 & \text{otherwise.} \end{cases}$$

Therefore

$$\int_{\mathbb{R}^d} |\nabla\varphi_m(x)|_2^2\, d\mu(x) = \int_{\{\varphi(x)\le m\}} |\nabla\varphi(x)|_2^2\, d\mu(x) \le \int_{\mathbb{R}^d} |\nabla\varphi(x)|_2^2\, d\mu(x) < \infty.$$

Moreover

$$\int_{\mathbb{R}^d} |\varphi(x)-\varphi_m(x)|\, d\mu(x) = \int_{\{|\varphi(x)|\ge m\}} |\varphi(x)-\varphi_m(x)|\, d\mu(x) \le 2\int_{\{|\varphi(x)|\ge m\}} |\varphi(x)|\, d\mu(x) \to 0$$

as $m \to \infty$ since $\varphi \in L^1(\mu;\mathbb{R}^d)$.

$$\int_{\mathbb{R}^d} |\nabla\varphi_m(x) - \nabla\varphi(x)|_2^2\, d\mu(x) = \int_{\{|\varphi(x)|>m\}} |\nabla\varphi(x)|_2^2\, d\mu(x) \to 0 \quad \text{as } m \to \infty,$$

since $\mathbb{1}_{\{|\varphi(x)|>m\}}(y) \to 0$ μ-a.e. as $m \to \infty$ and $\nabla\varphi \in L^2(\mu;\mathbb{R}^d)$. So we can suppose φ to be bounded in the assumption of the lemma.

(ii) We multiply the bounded function φ by $\eta(x/n)$ where $\eta \in C_c^1(\mathbb{R}^d)$, $0 \le \eta(x) \le 1$, $\eta(x) = 1$ for $|x|_2 \le 1$. Set $\varphi_n(x) := \varphi(x)\eta(x/n)$, $x \in \mathbb{R}^d$, $n \ge 1$. Then φ_n is globally Lipschitz continuous with compact support, hence $\varphi_n \in L^1(\mu;\mathbb{R}^d)$ and $\varphi_n \in L^2(\mu;\mathbb{R}^d)$ since $\nabla\varphi_n$ is bounded. Moreover, we have

$$\int_{\mathbb{R}^d} |\varphi(x) - \varphi_n(x)|\rho(x)\, dx = \int_{\mathbb{R}^d} |1 - \eta(x/n)|\, |\varphi(x)|\, \rho(x)\, dx \to 0 \quad \text{as } n \to \infty,$$

since $|\varphi|\rho \in L^1(\mathbb{R}^d)$ with respect to λ_d.

Similarly

$$\begin{aligned}\int_{\mathbb{R}^d} |\nabla\varphi(x) - \nabla\varphi_n(x)|^2\rho(x)\, dx &= \int_{\mathbb{R}^d}\Big|\Big(1-\rho\Big(\frac{x}{n}\Big)\Big)\nabla\varphi(x) - \frac{1}{n}\,\varphi(x)\nabla\rho\Big(\frac{x}{n}\Big)\Big|_2^2\rho(x)\, dx\\ &\le 2\int_{\mathbb{R}^d}\Big|1-\rho\Big(\frac{x}{n}\Big)\Big|^2\, |\nabla\varphi(x)|^2\rho(x)\, dx + \frac{2}{n^2}\,\|\varphi\|_\infty^2\,\|\nabla\rho\|_\infty^2 \to 0 \quad \text{as } n \to \infty.\end{aligned}$$

So we can assume φ globally Lipschitz continuous with compact support.

(iii) Finally taking the convolution with a Sobolev mollifier we will obtain a sequence $\varphi_{1/n} \in C_c^\infty(\mathbb{R}^d)$ satisfying

$$\int_{\mathbb{R}^d} |\varphi_{1/n} - \varphi|\, d\mu \to 0 \quad \text{and} \quad \int_{\mathbb{R}^d} |\nabla\varphi_{1/n} - \nabla\varphi|_2^2\, d\mu \to 0 \quad \text{as } n \to \infty. \qquad \square$$

2.2 The Benamou–Brenier formula

The aim of this section is to show that the sequence $\frac{1}{h_n}(r_{h_n} - \mathrm{id})$ considered in Section 2.1 tends *strongly* to $\boldsymbol{v}_t$ as $h_n \to 0$ whenever the metric derivative $|\dot\mu|(t)$ exists. Let $0 < T_1 < T_2$ be as in Section 2.1 and let $\bar t \in (T_1, T_2)$ be such that $|\dot\mu|(\bar t)$ exists. Notice that

$$\frac{1}{h_n^2}\int_{\mathbb{R}^d} |r_{h_n}(x) - x|_2^2\, d\mu_{\bar t}(x) = \frac{1}{h_n^2}\, W_2^2(\mu_{\bar t + h_n}, \mu_{\bar t})$$

which tends to $|\dot\mu|^2(\bar t)$ as $n \to \infty$.

On the other hand we have as a consequence of the weak convergence

$$\int_{\mathbb{R}^d} |\boldsymbol{v}_{\bar t}(x)|_2^2\, d\mu_{\bar t}(x) \le \varliminf_{n\to\infty} \frac{1}{h_n^2}\int_{\mathbb{R}^d} |r_{h_n}(x) - x|_2^2\, d\mu_{\bar t}(x) = \lim_{n\to\infty}\frac{1}{h_n^2}\int_{\mathbb{R}^d} |r_{h_n}(x) - x|_2^2\, d\mu_{\bar t}(x)$$

which implies

$$\|\boldsymbol{v}_{\bar t}\|^2_{L^2(\mu_{\bar t};\mathbb{R}^d)} \le |\dot\mu|^2(\bar t). \tag{2.11}$$

So it suffices to show that

$$|\dot\mu|^2(\bar t) \le \|\boldsymbol{v}_{\bar t}\|^2_{L^2(\mu_{\bar t};\mathbb{R}^d)}. \tag{2.12}$$

Indeed, the weak convergence together with the convergence of the norm of the sequence implies the strong convergence. Moreover, we get equality in (2.11) and (2.12). It turns out that (2.12) is a consequence of the Benamou–Brenier formula [4]. We recall this formula in a form which is more general than in [4] and less general than in [2], but sufficient for our purpose.

Theorem 2.2. *Let $a, b \in \mathbb{R}$ with $a < b$. Let*

$$[a,b] \times \mathbb{R}^d \ni (t,x) \mapsto \sigma_t(x) \in \mathbb{R},$$
$$[a,b] \times \mathbb{R}^d \ni (t,x) \mapsto \boldsymbol{w}_t(x) \in \mathbb{R}^d$$

be continuously differentiable functions such that

(i) *for every $t \in [a,b]$, σ_t is the density of a measure $\widehat\mu_t \in \mathcal{P}_2^a(\mathbb{R}^d)$,*

(ii) $[a,b] \ni t \mapsto \widehat\mu_t \in \mathrm{AC}([a,b]; (\mathcal{P}_2(\mathbb{R}^d), W_2(\cdot,\cdot)))$,

(iii) *for every $t \in [a,b]$, $\boldsymbol{w}_t \in L^2(\widehat\mu_t;\mathbb{R}^d)$ and*

$$\int_a^b \|\boldsymbol{w}_t\|^2_{L^2(\hat\mu_t;\mathbb{R}^d)}\, dt < \infty,$$

(iv) $\partial_t\sigma_t(x) + \mathrm{div}(\sigma_t \boldsymbol{w}_t)(x) = 0$, *$t \in [a,b]$, $x \in \mathbb{R}^d$.*

Then we have

$$W_2^2(\widehat\mu_a, \widehat\mu_b) \le (b-a) \int_a^b \|\boldsymbol{w}_t\|^2_{L^2(\hat\mu_t;\mathbb{R}^d)}\, dt. \tag{2.13}$$

Noticing that $\sigma_t(x) := \rho_t(x)$, $\boldsymbol{w}_t(x) := \boldsymbol{v}_t(x)$, $t \in [T_1, T_2]$, $x \in \mathbb{R}^d$, satisfy the assumptions of Theorem 2.2 we obtain for $0 < a = \bar t$, $b = \bar t + h$, $h > 0$, where $T_1 < \bar t < \bar t + h < T_2$

$$\frac{W_2^2(\mu_{\bar t}, \mu_{\bar t + h})}{h^2} \le \frac{1}{h} \int_{\bar t}^{\bar t + h} \|\boldsymbol{v}_t\|^2_{L^2(\mu_t;\mathbb{R}^d)}\, dt,$$

hence

$$|\dot\mu|^2(\bar t) \le \|\boldsymbol{v}_{\bar t}\|^2_{L^2(\mu_{\bar t};\mathbb{R}^d)}$$

This completes the proof of

$$\lim_{h\to 0} \int_{\mathbb{R}^d} \left| \frac{1}{h}(r_h(x) - x) - \boldsymbol{v}_{\bar t}(x) \right|_2^2 d\mu_{\bar t}(x) = 0, \tag{2.14}$$

$$\|\boldsymbol{v}_{\bar t}\|_{L^2(\mu_{\bar t};\mathbb{R}^d)} = |\dot\mu|(\bar t). \tag{2.15}$$

Sketch of the proof of Theorem 2.2. Let σ_t, $\boldsymbol{w}_t$ be as in Theorem 2.2. Consider the ODE

$$\begin{cases} \frac{d}{dt}X_t(x) = \boldsymbol{w}_t(X_t(x)), \\ X_a(x) = x \end{cases} \tag{2.16}$$

for every $x \in \mathbb{R}^d$ and $t \in [a,b]$.

Then, in view of Lemma 8.1.4 and Proposition 8.1.8 of [2], the characteristic system (2.16) admits a unique globally defined solution $X_t(x)$, $t \in [a,b]$, for $\widehat{\mu}_a$-a.a. $x \in \mathbb{R}^d$, and

$$\widehat{\mu}_t = (X_t)_{\#}\widehat{\mu}_a \quad \forall t \in [a,b]. \tag{2.17}$$

Therefore

$$\begin{aligned} W_2^2(\widehat{\mu}_a, \widehat{\mu}_b) &\le \int_{\mathbb{R}^d} |x - X_b(x)|_2^2 \, d\widehat{\mu}_a(x) = \int_{\mathbb{R}^d} |X_a(x) - X_b(x)|_2^2 \, d\widehat{\mu}_a(x) \\ &= (b-a)^2 \int_{\mathbb{R}^d} \left| \frac{1}{b-a} \int_a^b \frac{dX_s}{ds}(x) \, ds \right|_2^2 d\widehat{\mu}_a(x) \le (b-a) \int_{\mathbb{R}^d} \int_a^b |\boldsymbol{w}_s(X_s(x))|_2^2 \, ds \, d\widehat{\mu}_a(x) \\ &= (b-a) \int_a^b \int_{\mathbb{R}^d} |\boldsymbol{w}_s(y)|_2^2 \, d\widehat{\mu}_s(y) \, ds = (b-a) \int_a^b \|\boldsymbol{w}_t\|_{L^2(\widehat{\mu}_t;\mathbb{R}^d)}^2 \, dt. \quad \square \end{aligned}$$

For the sake of completeness we state a part of Theorem 8.3.1 of [2] which is a generalization of Theorem 2.2.

Theorem 2.3 ([2, Theorem 8.3.1, converse])**.** *Let $I := [a,b]$, $a, b \in \mathbb{R}$, $a < b$, and let $p > 1$. Let $I \ni t \mapsto \widehat{\mu}_t \in \mathcal{P}_p(\mathbb{R}^d)$ be narrowly continuous (i.e. with respect to the β metric) and let $I \times \mathbb{R}^d \ni (t,x) \mapsto \boldsymbol{w}_t(x)$ be Borel measurable such that the following holds:*

$$\int_a^b \left(\int_{\mathbb{R}^d} |\boldsymbol{w}_t(x)|_2^p \, d\widehat{\mu}_t(x) \right)^{1/p} dt < \infty, \tag{2.18}$$

$$\int_I \int_{\mathbb{R}^d} \big(\partial_t \varphi(t,x) + \langle \boldsymbol{w}_t(x), \nabla_x \varphi(t,x) \rangle \big) \, d\widehat{\mu}_t(x) \, dt = 0 \tag{2.19}$$

for every $\varphi \in C_c^\infty((a,b) \times \mathbb{R}^d)$ (weak solution to the continuity equation). Then $t \mapsto \widehat{\mu}_t \in \mathrm{AC}\big(I; (\mathcal{P}_p(\mathbb{R}^d), W_p(\cdot,\cdot))\big)$ and

$$|\dot{\mu}|(t) \le \|\boldsymbol{w}_t\|_{L^p(\widehat{\mu}_t;\mathbb{R}^d)} \quad \text{for } \lambda_1\text{-a.a. } t \in (a,b). \tag{2.20}$$

Remark. For $p = 2$, we have as a consequence of Theorem 2.3,

$$\begin{aligned} W_2(\widehat{\mu}_a, \widehat{\mu}_b) &\le \int_a^b |\dot{\mu}|(t) \, dt = (b-a) \frac{1}{b-a} \int_a^b |\dot{\mu}|(t) \, dt \\ &\le (b-a) \left(\frac{1}{b-a} \int_a^b |\dot{\mu}|^2(t) \, dt \right)^{1/2} \text{(by Jensen's inequality)} \\ &\le (b-a)^{1/2} \left(\int_a^b \|\boldsymbol{w}_t\|_{L^2(\widehat{\mu}_t;\mathbb{R}^d)}^2 \, dt \right)^{1/2}, \end{aligned}$$

by (2.20). Hence we recover the Benamou–Brenier inequality (2.13).

2.3 A representation formula for $\frac{1}{2}\frac{d}{dt}W_2^2(\mu_t,\sigma)$

Now we return to the study of the curve μ_t defined in (1.5) where $\mu_0 \in \mathcal{P}_2(\mathbb{R}^d)$. We already know that for $0 < T_1 < T_2$, $t \mapsto \mu_t$ belongs to $\mathrm{AC}\big([T_1,T_2];(\mathcal{P}_2(\mathbb{R}^d),W_2(\cdot,\cdot))\big)$, hence $t \mapsto W_2^2(\mu_t,\sigma)$ is absolutely continuous for every $\sigma \in \mathcal{P}_2(\mathbb{R}^d)$.

In the next proposition we give a formula for its derivative.

Proposition 2.1. *Let $\sigma \in \mathcal{P}_2(\mathbb{R}^d)$ and μ_t be as above. Let $t > 0$ be such that the metric derivative at t, $|\dot{\mu}|(t)$ exists and the derivative of $t \mapsto W_2^2(\mu_t,\sigma)$ at t exists. Then we have*

$$\frac{1}{2}\frac{d}{dt}W_2^2(\mu_t,\sigma) = \int_{\mathbb{R}^d} \big\langle x - r_{\mu_t}^{\sigma}(x), \boldsymbol{v}_t(x)\big\rangle \, d\mu_t(x) \tag{2.21}$$

where $r_{\mu_t}^{\sigma}$ is the unique optimal transport map from μ_t to σ and $\boldsymbol{v}_t(x)$ is given by (2.4).

Proof. Let $\sigma \in \mathcal{P}_2(\mathbb{R}^d)$, t as in the assumption and $h \in \mathbb{R}$ such that $h > -t$. By definition

$$\begin{aligned} W_2^2\big((\mathrm{id} + h\boldsymbol{v}_t)_{\#}\mu_t,\sigma\big) &\le \int_{\mathbb{R}^d} \big|x + h\boldsymbol{v}_t(x) - r_{\mu_t}^{\sigma}(x)\big|_2^2 \, d\mu_t(x) \\ = \int_{\mathbb{R}^d} \big|x - r_{\mu_t}^{\sigma}(x)\big|_2^2 \, d\mu_t(x) &+ 2h \int_{\mathbb{R}^d} \big\langle x - r_{\mu_t}^{\sigma}(x), \boldsymbol{v}_t(x)\big\rangle \, d\mu_t(x) + h^2 \int_{\mathbb{R}^d} |\boldsymbol{v}_t(x)|^2 \, d\mu_t(x) \\ &= W_2^2(\mu_t,\sigma) + 2h\mathrm{RHS} + h^2 \left\|\boldsymbol{v}_t\right\|^2_{L^2(\mu_t;\mathbb{R}^d)}, \end{aligned}$$

where RHS is the right hand side of (2.21).

Therefore we have for $h > -t$:

$h > 0$,

$$\frac{1}{h}[W_2^2(\mathrm{id} + h\boldsymbol{v}_{t\#}\mu_t,\sigma) - W_2^2(\mu_t,\sigma)] \le 2\mathrm{RHS} + h\|\boldsymbol{v}_t\|^2_{L^2(\mu_t;\mathbb{R}^d)},$$

$h > 0$,

$$\frac{1}{h}[W_2^2(\mathrm{id} + h\boldsymbol{v}_{t\#}\mu_t,\sigma) - W_2^2(\mu_t,\sigma)] \ge 2\mathrm{RHS} + h\|\boldsymbol{v}_t\|^2_{L^2(\mu_t;\mathbb{R}^d)}.$$

If we can show that

$$\lim_{t \underset{>}{\to} 0} \frac{1}{h}\big(W_2^2((\mathrm{id} + h\boldsymbol{v}_t)_{\#}\mu_t,\sigma) - W_2^2(\mu_t,\sigma)\big) = \lim_{t \underset{<}{\to} 0} \frac{1}{h}\big(W_2^2((\mathrm{id} + h\boldsymbol{v}_t)_{\#}\mu_t,\sigma) - W_2^2(\mu_t,\sigma)\big) =: L$$

exist, then we get $\frac{1}{2}L = \mathrm{RHS}$.

In Section 2.2 we have shown that

$$\lim_{h\to 0} \frac{1}{h}\big(r_{\mu_t}^{\mu_{t+h}} - \mathrm{id}\big) = v_t \quad \text{in } L^2(\mu_t;\mathbb{R}^d).$$

Therefore for $h \neq 0$,

$$\frac{1}{h^2}\, W_2^2\big(\mu_{t+h}, (\mathrm{id} + hv_t)_{\#}\mu_t\big) \le \frac{1}{h^2}\int_{\mathbb{R}^d} \big|r_{\mu_t}^{\mu_{t+h}} - (\mathrm{id} + hv_t)\big|_2^2 \, d\mu_t \to 0 \quad \text{as } h \to 0,$$

hence

$$\lim_{h\to 0} \frac{W_2(\mu_{t+h}, (\mathrm{id} + hv_t)_{\#}\mu_t)}{|h|} = 0.$$

Next we estimate, for $0 < |h| < t \wedge 1$:

$$\begin{aligned}
&\Big|\frac{1}{h}\big(W_2^2\big((\mathrm{id}+hv_t)_{\#}\mu_t,\sigma\big) - W_2^2(\mu_{t+h},\sigma)\big)\Big| \\
&\quad = \big(W_2\big((\mathrm{id}+hv_t)_{\#}\mu_t,\sigma\big) + W_2(\mu_{t+h},\sigma)\big)\cdot\frac{1}{|h|}\,\big|W_2\big((\mathrm{id}+hv_t)_{\#}\mu_t,\sigma\big) - W_2(\mu_{t+h},\sigma)\big| \\
&\qquad\qquad \le \big(W_2\big((\mathrm{id}+hv_t)_{\#}\mu_t,\sigma\big) + W_2(\mu_{t+h},\sigma)\big)\cdot\frac{1}{|h|}\,W_2(\mathrm{id}+hv_t)_{\#}\mu_t,\mu_{t+h}\big).
\end{aligned}$$

We have $W_2^2\big((\mathrm{id}+hv_t)_{\#}\mu_t,\sigma\big) \le W_2^2(\mu_t,\sigma) + 2\mathrm{RHS} + \|v\|^2_{L^2(\mu_t;\mathbb{R}^d)}$ and $W_2^2(\mu_{t+h},\sigma)$ is bounded since μ_t is absolutely continuous, therefore

$$\lim_{h\to 0}\frac{1}{|h|}\,\big|W_2^2\big((\mathrm{id}+hv_t)_{\#}\mu_t,\sigma\big) - W_2(\mu_{t+h},\sigma)\big| = 0.$$

Since the derivative of $t \mapsto W_2^2(\mu_t,\sigma)$ exists at t, we obtain that the above limits exist and are equal to L with $L = \frac{d}{dt}W_2^2(\mu_t,\sigma)$. This completes the proof of (2.21). □

As a consequence of (2.21) we have that for every $\sigma \in \mathcal{P}_2(\mathbb{R}^d)$,

$$\frac{1}{2}\frac{d}{dt}W_2^2(\mu_t,\sigma) = \int_{\mathbb{R}^d}\Big\langle x - r^{\sigma}_{\mu_t}(x), -\frac{\beta}{2}\,\frac{\nabla\rho_t(x)}{\rho_t(x)}\Big\rangle\,d\mu_t(x) \tag{2.22}$$

for λ_1-a.a. $t \in (0,\infty)$.

In order to achieve the goal mentioned in the introduction, we have to show that the right hand side of (2.22) satisfies

$$\mathrm{RHS} \le \phi(\sigma) - \phi(\mu_t) \tag{2.23}$$

for some l.s.c. functional $\phi : (\mathcal{P}_2(\mathbb{R}^d), W_2(\cdot,\cdot)) \to (-\infty,+\infty]$.

This will be done in Section 3.

We conclude this section by stating a theorem which shows that formula (2.2) holds in a more general situation.

Theorem 2.4 ([2, Theorem 8.3.1, first part, and Theorem 8.4.7])**.** *Let $I = [a,b]$ with $a,b \in \mathbb{R}$, $a < b$, and let $I \ni t \mapsto \widehat{\mu}_t \in \mathcal{P}_2(\mathbb{R}^d)$ be absolutely continuous with respect to the metric $W_2(\cdot,\cdot)$. Let $|\dot{\widehat{\mu}}|(\cdot)$ be its metric derivative, defined in Theorem 1.2. Then there exists a Borel vector field*

$$w : I \times \mathbb{R}^d \ni (t,x) \mapsto w_t(x) \in \mathbb{R}^d \quad \textit{such that}$$

- (i) $\int_I \int_{\mathbb{R}^d} |w_t(x)|_2^2\, d\widehat{\mu}_t\, dt < \infty$,
- (ii) $\int_{\mathbb{R}^d} |w_t(x)|_2^2\, d\widehat{\mu}_t\, dt = |\dot{\widehat{\mu}}|^2(t)$ *a.e. in* (a,b),
- (iii) w_t *satisfies* (2.19),
- (iv) $w_t \in Tan_{\widehat{\mu}_t}\mathcal{P}_2(\mathbb{R}^d)$ (*see Definition 2.1*) *and is uniquely defined a.e. in* (a,b),
- (v) *for every* $\sigma \in \mathcal{P}_2(\mathbb{R}^d)$ *and for every* $t > 0$ *such that*: $|\dot{\widehat{\mu}}|(\cdot)$ *exists,* $|\dot{\widehat{\mu}}|(t) = \|w_t\|_{L^2(\widehat{\mu}_t;\mathbb{R}^d)}$ *and* $\frac{1}{2}\frac{d}{dt}W_2^2(\widehat{\mu}_t,\sigma)$ *exists, we have*

$$\frac{1}{2}\frac{d}{dt}W_2^2(\widehat{\mu}_t,\sigma) = \int_{\mathbb{R}^d}\langle x - r^{\sigma}_{\widehat{\mu}_t}(x), \boldsymbol{w}_t(x)\rangle\, d\widehat{\mu}_t(x) \tag{2.24}$$

where $r^{\sigma}_{\widehat{\mu}_t}$ is the unique optimal transport map defined in Theorem 2.1.

3 The functional $\phi(\rho) = \int_{\mathbb{R}^d} \rho \log \rho \, dx$

The aim of this section is to show that there exists a l.s.c. functional

$$\phi : \big(\mathcal{P}_2(\mathbb{R}^d), W_2(\cdot,\cdot)\big) \to (-\infty, +\infty]$$

such that for every $\sigma \in D(\phi)$ and every $t > 0$ we have

$$\int_{\mathbb{R}^d} \left\langle x - r^{\sigma}_{\mu_t}(x), -\frac{\beta}{2}\,\frac{\nabla \rho_t(x)}{\rho_t(x)} \right\rangle d\mu_t \le \phi(\sigma) - \phi(\mu_t), \tag{3.1}$$

where μ_t is defined in (1.5).

As a consequence the map $(0,\infty) \ni t \mapsto \mu_t \in \mathcal{P}_2(\mathbb{R}^d)$ satisfies (EVI) with this functional and with $X = \mathcal{P}_2(\mathbb{R}^d)$ equipped with the metric $W_2(\cdot,\cdot)$, and with $\alpha = 0$.

In view of the uniqueness of solutions to the initial value problem for (EVI) (see [8]) and the contractivity of $\{S_\beta(t)\}_{t\ge 0}$ with respect to $W_2(\cdot,\cdot)$ it is possible to choose $\alpha = 0$ in (EVI).

Since the semigroup $\{S_\beta(t)\}_{t\ge 0}$ is defined on the whole of $\mathcal{P}_2(\mathbb{R}^d)$, the domain of ϕ should be dense. In Section 3.1 we invoke a result of Daneri and Savaré [12] to show that if such functional ϕ exists then it has to be "displacement convex" in the sense of McCann [17].

In the second part of Section 3.1 we show that if the functional ϕ is of the form $\phi(\mu) = \int_{\mathbb{R}^d} F(\rho(x))\, dx$, for $\mu \in \mathcal{P}_2^a(\mathbb{R}^d)$ with $\mu = \rho \cdot \lambda_d$, where the function $F : [0,\infty) \to \mathbb{R}$ is continuous, satisfies $F(0) = 0$, is continuously differentiable on $(0,\infty)$, $\lim\limits_{s\to 0} sF'(s) = 0$, and $\phi(\rho_t) < \infty$ for every $t > 0$ and $\mu_0 \in \mathcal{P}_2(\mathbb{R}^d)$, then $F(s) = \frac{\beta}{2}\, s \log s$, $s > 0$. In Section 3.2 we consider the functionals ϕ_m induced by the approximation $F_m(s) = \frac{1}{m-1}(s^m - s)$, $s > 0$, $m > 1$, and in Section 3.3 we show that the functional ϕ satisfies (3.1). Finally, using the decreasingness of $t \mapsto \phi(\mu_t)$ we prove that μ_t satisfies (EVI).

From now on when we consider the space $\mathcal{P}_2(\mathbb{R}^d)$ we always assume that it is equipped with the metric $W_2(\cdot,\cdot)$.

3.1 Displacement convexity

Following Daneri–Savaré [12] we have following proposition.

Proposition 3.1. *Suppose that there exists a l.s.c. functional $\phi : \mathcal{P}_2(\mathbb{R}^d) \to (-\infty, +\infty]$ such that for every $\mu_0 \in \mathcal{P}_2(\mathbb{R}^d)$, the map $(0,\infty) \ni t \mapsto S_\beta(t)\mu_0 \in \mathcal{P}_2(\mathbb{R}^d)$ satisfies* (EVI) *with $\alpha = 0$ and the functional ϕ. Then for every* (*geodesic*) *curve $\gamma : [0,1] \to \mathcal{P}_2(\mathbb{R}^d)$ satisfying*

$$W_2(\gamma^r, \gamma^s) = |r - s|\, W_2(\gamma^0, \gamma^1), \quad 0 \le r, s \le 1, \tag{3.2}$$

and

$$\phi(\gamma^0), \phi(\gamma^1) < \infty, \tag{3.3}$$

we have

$$\phi(\gamma^s) \le (1-s)\phi(\gamma^0) + s\phi(\gamma^1) \quad \forall s \in (0,1). \tag{3.4}$$

Proof. Let $s, t_1, t_2 \in (0,1)$ with $t_1 < t_2$. We have

$$\frac{1}{2}\, W_2^2(S_\beta(t_2)\gamma^s, \gamma^i) - \frac{1}{2}\, W_2^2(S_\beta(t_1)\gamma^s, \gamma^i) \le (t_2 - t_1)\big[\phi(\gamma^i) - \phi(S_\beta(t_2)\gamma^s)\big], \quad i = 1, 2,$$

by (EVI).

We multiply this inequality by $(1-s)$ for $i = 0$ and by s for $i = 1$, and add. We obtain

$$\begin{aligned} &\frac{1}{2}\big[(1-s)W_2^2(S_\beta(t_2)\gamma^s, \gamma^0) + sW_2^2(S_\beta(t_2)\gamma^s, \gamma^1)\big] \\ &\qquad - \frac{1}{2}\big[(1-s)W_2^2(S_\beta(t_1)\gamma^s, \gamma^0) + sW_2^2(S_\beta(t_1)\gamma^s, \gamma^1)\big] \\ &\qquad\qquad \le (t_2 - t_1)\big[(1-s)\phi(\gamma^0) + s\phi(\gamma^1) - \phi(S_\beta(t_2)\gamma^s)\big]. \end{aligned}$$

Using the right-continuity of $S_\beta(t)\mu_0$ at $t = 0$, we obtain, letting $t_1 \to 0$,

$$\begin{aligned} &\frac{1}{2}\big[(1-s)W_2^2(S_\beta(t_2)\gamma^s, \gamma^0) + sW_2^2(S_\beta(t_2)\gamma^s, \gamma^1)\big] \\ &\quad - \frac{1}{2}\big[(1-s)W_2^2(\gamma^s, \gamma^0) + sW_2^2(\gamma^s, \gamma^1)\big] \le t_2\big[(1-s)\phi(\gamma^0) + s\phi(\gamma^1) - \phi(S_\beta(t_2)\gamma^s)\big]. \end{aligned} \tag{3.5}$$

Next we claim that the left hand side of (3.5) is nonnegative. Indeed, using the inequality

$$(1-s)a^2 + sb^2 \ge s(1-s)(a+b)^2, \quad a, b \in \mathbb{R}, \quad s \in (0,1), \tag{3.6}$$

we get

$$\begin{aligned} &(1-s)W_2^2(S_\beta(t_2)\gamma^s, \gamma^0) + sW_2^2(S_\beta(t_2)\gamma^s, \gamma^1) \\ &\qquad \ge s(1-s)\big[W_2(S_\beta(t_2)\gamma^s, \gamma^0) + W_2(S_\beta(t_2)\gamma^s, \gamma^1)\big]^2 \ge s(1-s)W_2^2(\gamma^0, \gamma^1), \end{aligned}$$

using the triangle inequality.

On the other hand, by using (3.2), we have

$$\begin{aligned} &(1-s)W_2(\gamma^s, \gamma^0) + sW_2(\gamma^s, \gamma^1) \\ &\qquad = [(1-s)s^2 + s(1-s)^2]W_2^2(\gamma^0, \gamma^1) = s(1-s)W_2^2(\gamma^0, \gamma^1). \end{aligned}$$

This establishes the claim. Therefore, after dividing by $t_2 > 0$, we obtain

$$\phi(S_\beta(t_2)\gamma^s) \le (1-s)\phi(\gamma^0) + s\phi(\gamma^1).$$

Using again the right continuity of $S_\beta(t_2)\gamma^s$ at $t = 0$ and the lower semicontinuity of ϕ we obtain (3.4). □

Remark. Observe that in (3.2) equality can be replaced by the inequality $\le$. Indeed, suppose that the curve $\gamma : [0,1] \to \mathcal{P}_2(\mathbb{R}^d)$ satisfies

$$W_2(\gamma^r, \gamma^s) \le |r - s|\, W_2(\gamma^0, \gamma^1), \quad 0 \le r, s \le 1, \tag{3.7}$$

then if for some $0 \le r < s \le 1$ strict inequality in (3.7) occurs, then we get

$$\begin{aligned} &W_2(\gamma^0, \gamma^1) \le W_2(\gamma^0, \gamma^r) + W_2(\gamma^r, \gamma^s) + W_2(\gamma^s, \gamma^1) \\ &\qquad < (r + (s-r) + 1 - s)W_2(\gamma^0, \gamma^1) = W_2(\gamma^0, \gamma^1), \end{aligned}$$

a contradiction.

The next proposition provides a geodesics between a measure $\mu \in \mathcal{P}_2^a(\mathbb{R}^d)$ and $\sigma \in \mathcal{P}_2(\mathbb{R}^d)$.

Proposition 3.2. *Let $\mu \in \mathcal{P}_2^a(\mathbb{R}^d)$ and $\sigma \in \mathcal{P}_2(\mathbb{R}^d)$. Let r_μ^σ be the optimal transport map between μ and σ. Set*

$$\gamma^t := ((1-t)\,\mathrm{id} + tr_\mu^\sigma)_\#\mu, \quad t \in [0,1].$$

Then γ^t satisfies (3.2).

Proof. In view of the preceding remark it is sufficient to prove (3.7). Let $0 \le t < s \le 1$. Notice that $\big((1-t)\,\mathrm{id} + tr_\mu^\sigma, (1-s)\,\mathrm{id} + sr_\mu^\sigma\big)_\#\mu \in \Gamma(\gamma^t, \gamma^s)$. Hence

$$\begin{aligned} W_2^2(\gamma^t,\gamma^s) &\le \int_{\mathbb{R}^d\times\mathbb{R}^d} |x_1 - x_2|_2^2\, d\big((1-t)\,\mathrm{id} + tr_\mu^\sigma, (1-s)\,\mathrm{id} + sr_\mu^\sigma\big)_\#\mu(x_1,x_2) \\ &= \int_{\mathbb{R}^d} \big|(1-t)x + tr_\mu^\sigma(x) - (1-s)x - sr_\mu^\sigma(x)\big|_2^2\, d\mu(x) \\ &= \int_{\mathbb{R}^d} \big|(s-t)x - (s-t)r_\mu^\sigma(x)\big|_2^2\, d\mu(x) = |s-t|^2 W_2^2(\mu,\sigma). \qquad \square \end{aligned}$$

McCann [17] introduced the notion of *displacement convexity* and proved that many important functionals $\phi : \mathcal{P}_2(\mathbb{R}^d) \to (-\infty, +\infty]$ satisfy

$$\phi\big(((1-t)\,\mathrm{id} + tr_\mu^\sigma)_\#\mu\big) \le (1-t)\phi(\mu) + t\phi(\sigma), \quad t \in [0,1], \tag{3.8}$$

for $\mu \in \mathcal{P}_2^a(\mathbb{R}^d)$, $\sigma \in \mathcal{P}_2^a(\mathbb{R}^d)$.

Such functionals are called *displacement convex*.

This notion can be generalized. In particular, if $\mu \in \mathcal{P}_2(\mathbb{R}^d) \setminus \mathcal{P}_2^a(\mathbb{R}^d)$ then optimal transport maps should be replaced by optimal transport plans [17]. Moreover, the notion of convexity can be extended to the notion of λ-convexity [2]. However, for our purpose the notion of displacement convexity (3.8) will be sufficient.

In the remaining part of this section we show that if the functional ϕ in (3.1) is of the form $\phi(\mu) := \int_{\mathbb{R}^d} F(\rho(x))\,dx$ where $\mu \in \mathcal{P}_2^a(\mathbb{R}^d)$ with $\mu = \rho \cdot \lambda_d$, for some function $F : [0,\infty) \to \mathbb{R}$ satisfying certain mild conditions, then $F(s) = \frac{\beta}{2} s \log s + Cs$, $s > 0$, where C is any real number.

Lemma 3.1. *Let $F : [0,\infty) \to \mathbb{R}$ be continuous and satisfy $F(0) = 0$, $F|_{(0,\infty)} \in C^1(0,\infty)$ and $\lim_{\rho\to 0} \rho F'(\rho) = 0$. Suppose $\int_{\mathbb{R}^d} |F(\rho_t(x))|\,dx < \infty$ for every $t > 0$ and every ρ_t defined in* (1.6) *with arbitrary $\mu_0 \in \mathcal{P}_2(\mathbb{R}^d)$. Then the following holds.*

(i) *For every $\varphi \in C_c^\infty(\mathbb{R}^d)$ not identically zero there exists $\bar\varepsilon > 0$ such that for every $\varepsilon \in (-\bar\varepsilon, \bar\varepsilon)$, the map $r := \mathrm{id} + \varepsilon\nabla\varphi$ is the optimal transport map $r_{\mu_t}^\sigma$ where $\sigma := r_\#\mu_t$ and $\mu_t := \rho_t \cdot \lambda_d$. Moreover, $\sigma \in \mathcal{P}_2^a(\mathbb{R}^d)$, with density ρ_t^ε given by*

$$\rho_t^\varepsilon(y) := \frac{\rho(\cdot)}{\det(I + \varepsilon J(\nabla\varphi))} \circ (\mathrm{id} + \varepsilon\nabla\varphi)^{-1}(y) \tag{3.9}$$

for every $y \in \mathbb{R}^d$.

(ii) $\int_{\mathbb{R}^d} \big|F(\rho_t^\varepsilon(y))\big|\,dy < \infty$ *for every $t > 0$, $0 \le |\varepsilon| < \bar\varepsilon$ and ρ_t defined in* (1.6).

(iii) *If* (3.1) *holds for every* ρ_t, ρ_t^ε *defined above, for every* $t > 0$ *where*

$$\phi(\mu_t) := \int_{\mathbb{R}^d} F(\rho_t(x))\, dx \quad \textit{and} \quad \phi(\sigma) := \int_{\mathbb{R}^d} F(\rho_t^\varepsilon(y))\, dy,$$

there exists $C \in \mathbb{R}$ *such that* $F(s) = \frac{\beta}{2} s \log s + Cs$ *for every* $s > 0$.

Remark. The assumption $F(0) = 0$ is necessary for $\int_{\mathbb{R}^d} |F(\rho_t(x)|\, dx$ to be finite.

Proof of Lemma 3.1. (i) We choose $\bar\varepsilon$ such that $\bar\varepsilon[\nabla\varphi]_{\mathrm{Lip}} < 1$. Then $r : \mathbb{R}^d \to \mathbb{R}^d$ is a bijection, $r \in C^1(\mathbb{R}^d, \mathbb{R}^d)$ as well as r^{-1} by the contraction mapping theorem and satisfies the assumptions of Lemma A2.1 of Appendix 2. Therefore $\sigma := r_{\#}\mu_t \in \mathcal{P}^a(\mathbb{R}^d)$ with density given by (3.9). Moreover

$$\begin{aligned}\int_{\mathbb{R}^d} |x|_2^2\, d\sigma(x) &= \int_{\mathbb{R}^d} |x|_2^2\, dr_{\#}\mu_t(x) \\ &= \int_{\mathbb{R}^d} |r(x)|_2^2\, d\mu_t(x) \le 2\int_{\mathbb{R}^d} |x|_2^2\, d\mu_t(x) + 2\varepsilon^2 \int_{\mathbb{R}^d} |\nabla\phi(x)|_2^2\, d\mu_t(x) < \infty,\end{aligned}$$

since $\nabla\phi \in C_c^\infty(\mathbb{R}^d, \mathbb{R}^d)$. So $\sigma \in \mathcal{P}_2^a(\mathbb{R}^d)$. Observe that r is monotone, i.e.

$$\langle r(x) - r(\widehat{x}), x - \widehat{x}\rangle \ge 0, \quad x, \widehat{x} \in \mathbb{R}^d,$$

by the choice of ε. Therefore the function $\psi(x) := \frac{1}{2}|x|^2 + \varepsilon\varphi(x)$, satisfying $\nabla\psi = \mathrm{id} + \varepsilon\nabla\varphi$ is convex. It follows from Lemma A3.1 in Appendix 3 that $r = r_{\mu_t}^\sigma$.

(ii) Since $\nabla\varphi \in C_c^\infty(\mathbb{R}^d, \mathbb{R}^d)$, it follows from (3.9) that there exists $R > 0$ such that $\rho_t(x) = \rho_t^\varepsilon(x)$, $|x|_2 \ge R$. Therefore

$$\int_{|x|_2 \ge R} |F(\rho_t^\varepsilon(x))|\, dx = \int_{|x|_2 \ge R} |F(\rho_t(x))|\, dx < \infty.$$

Since ρ_t^ε is continuous as well as $F(\rho_t^\varepsilon)$,

$$\int_{|x|_2 \le R} |F(\rho_t^\varepsilon(x))|\, dx < \infty.$$

(iii) Since $\rho_t(x)$, $\rho_t^\varepsilon(x)$ are positive, the maps

$$(-\bar\varepsilon, \bar\varepsilon) \times \mathbb{R}^d \ni (\varepsilon, x) \mapsto F(\rho_t^\varepsilon(x)), F(\rho_t(x))$$

(for fixed $t > 0$) are continuously differentiable. From parts (i), (ii) and (3.1) we obtain

$$\overline{\lim_{\varepsilon\downarrow 0}} \frac{\phi(\sigma) - \phi(\mu_t)}{\varepsilon} \le \int_{\mathbb{R}^d} \left\langle \nabla\varphi(x), \frac{\beta}{2}\frac{\nabla\rho_t(x)}{\rho_t(x)} \right\rangle d\mu_t(x) \le \underline{\lim_{\varepsilon\uparrow 0}} \frac{\phi(\sigma) - \phi(\mu_t)}{\varepsilon}. \tag{3.10}$$

Next we show that $\lim\limits_{\varepsilon\to 0} \frac{\phi(\sigma)-\phi(\mu_t)}{\varepsilon}$ exists. To this end we rewrite $\phi(\sigma) := \int_{\mathbb{R}^d} F(\rho_t^\varepsilon(x))\, dx$ as

$$\int_{\mathbb{R}^d} F\Big(\frac{\rho_t(x)}{\det(I + \varepsilon J(\nabla\varphi(x)))}\Big) \det(I + \varepsilon J\nabla\varphi(x))\, dx$$

by making use of the change of variable formula. Setting $A(\varepsilon)(x) := \det(I + \varepsilon J(\nabla\varphi(x)))$, $\varepsilon \in (-\bar\varepsilon, \bar\varepsilon)$, $x \in \mathbb{R}^d$, and $G(\varepsilon, x) := F\big(\frac{\rho_t(x)}{\det A(\varepsilon)(x)}\big) \det A(\varepsilon)(x)$, we have $G(0, x) = F(\rho_t(x))$ and

$$\frac{\phi(\sigma) - \phi(\mu_t)}{\varepsilon} = \int_{\mathbb{R}^d} \frac{1}{\varepsilon}\,[G(\varepsilon, x) - G(0, x)]\, dx, \quad \varepsilon \ne 0.$$

Let $R > 0$ be such that $\nabla\varphi(x) = 0$ for $|x| \geq R$. For these x's $G(\varepsilon, x) = G(0, x)$, hence

$$\frac{\phi(\sigma) - \phi(\mu_t)}{\varepsilon} = \int_{B(0,R)} \frac{1}{\varepsilon} \left[G(\varepsilon, x) - G(0, x)\right] dx.$$

Since $G : (-\overline{\varepsilon}, \overline{\varepsilon}) \times \mathbb{R}^d$ is C^1, we have for $0 < |\varepsilon| \leq \overline{\varepsilon}$, $|x|_2 \leq R$:

$$\frac{1}{|\varepsilon|} \left|G(\varepsilon, x) - G(0, x)\right| \leq \sup_{\substack{0 \leq |s| \leq \overline{\varepsilon} \\ y \in \overline{B(0,x)}}} \left|\frac{\partial G}{\partial \varepsilon}(s, y)\right| < \infty.$$

Therefore by the Lebesgue dominated convergence theorem we obtain

$$\lim_{\varepsilon \to 0} \frac{\phi(\sigma) - \phi(\mu_t)}{\varepsilon} = \int_{\mathbb{R}^d} \frac{\partial G}{\partial \varepsilon}(0, x)\, dx.$$

Using the fact that $A(0)(x) = 1$, we get

$$\frac{\partial G}{\partial \varepsilon}(0, x) = F'(\rho_t(x))(-1) \frac{\partial}{\partial \varepsilon} A(\varepsilon, x)|_{\varepsilon=0} \cdot \rho_t(x) + F(\rho_t(x))\, \frac{\partial}{\partial \varepsilon} A(\varepsilon, x)|_{\varepsilon=0}\,.$$

Noticing that $\frac{\partial}{\partial \varepsilon} A(\varepsilon, x)|_{\varepsilon=0} = \operatorname{Tr} J(\nabla\varphi(x))$, we obtain

$$\frac{\partial G}{\partial \varepsilon}(0, x) = \big[F(\rho_t(x)) - \rho_t(x) F'(\rho_t(x))\big] \cdot \operatorname{Tr} J(\nabla\varphi(x)).$$

Setting

$$L_F(\rho) := \rho F'(\rho) - F(\rho) \quad \text{for } \rho > 0, \tag{3.11}$$

we have

$$\lim_{\varepsilon \to 0} \frac{\phi(\sigma) - \phi(\mu_t)}{\varepsilon} = -\int_{\mathbb{R}^d} L_F(\rho_t(x)) \operatorname{div} \nabla\varphi(x)\, dx. \tag{3.12}$$

Combining (3.10) and (3.12) we get

$$-\int_{\mathbb{R}^d} L_F(\rho_t(x)) \Delta\varphi(x)\, dx = \int_{\mathbb{R}^d} \left\langle \nabla\varphi(x), \frac{\beta}{2} \nabla\rho_t(x) \right\rangle dx = -\int_{\mathbb{R}^d} \frac{\beta}{2}\, \rho_t(x) \Delta\varphi(x)\, dx.$$

Hence

$$\int_{\mathbb{R}^d} \Big(L_F(\rho_t(x)) - \frac{\beta}{2}\, \rho_t(x)\Big) \Delta\varphi(x) = 0$$

for every $\varphi \in C_c^\infty(\mathbb{R}^d)$. Noticing that $L_F(\rho_t) - \frac{\beta}{2}\rho_t \in C(\mathbb{R}^d)$, we obtain by Weyl's lemma that $L_F(\rho_t) - \frac{\beta}{2}\rho_t$ is $C^2(\mathbb{R}^d)$ and harmonic, that is

$$\Delta\Big(L_F(\rho_t(x)) - \frac{\beta}{2}\, \rho_t(x)\Big) = 0, \quad x \in \mathbb{R}^d.$$

Moreover, since $F \in C(0, \infty)$, $F(0) = 0$, $F \in C^1(0, \infty)$ and $\lim_{s \to 0} sF'(s) = 0$, we see that $L_F(\rho_t(x)) - \frac{\beta}{2}\rho_t(x) \to 0$ as $|x|_2 \to \infty$. Therefore $L_F(\rho_t(x)) = \frac{\beta}{2}\rho_t(x)$ for all $x \in \mathbb{R}^d$ as a consequence of Liouville's theorem for harmonic functions.

Moreover, since $\{\rho_t(x) : t > 0, x \in \mathbb{R}^d, \mu_0 \in \mathcal{P}_2(\mathbb{R}^d)\} = \mathbb{R}$, we have

$$L_F(s) := sF'(s) - F(s) = \frac{\beta}{2}\, s, \quad \text{for all } s > 0.$$

Since the C^∞ functions $h : (0,\infty) \to \mathbb{R}$ given by $h(s) = \frac{\beta}{2}\, s \log s + Cs$ where $C = h(1)$, satisfy the differential equation

$$sh'(s) - h(s) = \frac{\beta}{2}\, s, \quad h(1) = C,$$

by uniqueness of the initial (resp. final) value problem we can conclude that $F(s) = \frac{\beta}{2}\, s \log s + Cs$ for some $C \in \mathbb{R}$. This completes the proof of Lemma 3.1. □

In the next section we shall study the functional

$$\phi_m(\mu) := \int_{\mathbb{R}^d} \frac{1}{m-1}(\rho(x)^m - \rho(x))\, dx$$

for $\mu \in \mathcal{P}_2^a(\mathbb{R}^d)$ with $\mu = \rho \cdot \lambda_d$, and $m > 1$. Notice that

$$\lim_{m \downarrow 1} \frac{1}{m-1}(s^m - s) = s \log s, \quad \text{for } s > 0.$$

3.2 Subdifferential of ϕ_m

The aim of this section is to prove the following

Proposition 3.3. *Let $m > 1$, $\rho := \rho_t$ defined in (1.6) and let $\sigma \in \mathcal{P}_2^a(\mathbb{R}^d)$ with density $\widetilde{\rho} \in C_c^\infty(\mathbb{R}^d)$. Then we have*

$$\int_{\mathbb{R}^d} \widetilde{\rho}(x)^m\, dx - \int_{\mathbb{R}^d} \rho(x)^m\, dx \geq \int_{\mathbb{R}^d} m(m-1)\rho(x)^{m-1} \left\langle \nabla\rho(x), r_\mu^\sigma(x) - x \right\rangle dx. \tag{3.13}$$

Remark. Notice that if $F_m(s) := s^m$ for $m > 1$ and $s \geq 0$, then

$$m(m-1)\rho(x)^{m-1}\nabla\rho(x) = \nabla L_{F_m}(\rho(x))$$

where $L_{F_m}(s) := sF_m'(s) - F_m(s)$, $s \geq 0$.

Proof. First we observe that all integrals are well defined. Indeed, both $\widetilde{\rho}$ and ρ are continuous, nonnegative, bounded and integrable, therefore the first two integrals are well defined. Since $\rho(x) > 0$, $x \in \mathbb{R}^d$, $\rho \in C^1(\mathbb{R}^d)$, $\frac{\nabla\rho(x)}{\rho(x)}$ is well defined. So we can rewrite the last integral as

$$m(m-1)\int_{\mathbb{R}^d} \rho(x)^{m-1} \left\langle \frac{\nabla\rho(x)}{\rho(x)}, r_\mu^\sigma(x) - x \right\rangle \rho(x)\, dx. \tag{3.14}$$

If we set $\mu = \rho \cdot \lambda_d$, the integral is well defined since $\frac{\nabla\rho}{\rho} \in L^2(\mu;\mathbb{R}^d)$ by (2.10) as well as id, and ρ is bounded.

The first part of the proof consists of using the "displacement convexity" of the functional $\rho \mapsto \int_{\mathbb{R}^d} \rho(x)^m\, dx$, property discovered by McCann in his fundamental paper [17]. To this end we consider the optimal transport map r_μ^σ between μ and σ. Its existence and uniqueness as an element of $L^2(\mu;\mathbb{R}^d)$ is guaranteed by Theorem 2.1(ii) since $\mu \in \mathcal{P}_2^a(\mathbb{R}^d)$

and $\sigma \in \mathcal{P}_2(\mathbb{R}^d)$. Since $\widetilde{\rho} \in C_c^\infty(\mathbb{R}^d)$ it follows from Theorem 2.1(iii) that there exists $\psi_1 \in \mathrm{Lip}(\mathbb{R}^d)$ convex such that $r_\mu^\sigma = \nabla\psi_1$ λ_d-a.e. On the other hand, by Theorem A3.2(iii) in Appendix 3, there exists $\psi_2 : \mathbb{R}^d \to \mathbb{R}$ convex, locally Lipschitz continuous and a set N λ_d-negligible such that for every $x \in \mathbb{R}^d \setminus N$, $\nabla\psi_2(x)$ exists, $\nabla^2\psi_2(x)$ exists and is positive definite and finally ψ_2 is such that $r_\mu^\sigma = \nabla\psi_2$ λ_d-a.e. Since $\psi := \psi_1 - \psi_2$ is locally Lipschitz continuous and $\nabla\psi(x) = 0$ λ_d-a.e. in $\mathbb{R}^d$, it is constant. We can choose this constant to be 0, hence $\psi = \psi_1 = \psi_2$. By Theorem A3.2(b) we get

$$\int_{\mathbb{R}^d} \widetilde{\rho}(x)^m \, dx = \int_{\mathbb{R}^d\setminus N} \rho(x)^m \big(\det \nabla^2\psi(x)\big)^{1-m} \, dx. \tag{3.15}$$

Next we use the "McCann interpolant" where we replace the function ψ by the function

$$\psi_s(x) := (1-s)\tfrac{1}{2}|x|_2^2 + s\psi(x), \quad s \in [0,1],\ x \in \mathbb{R}^d.$$

The functions ψ_s are still convex, Lipschitz continuous. Moreover, for $x \in \mathbb{R}^d \setminus N$: $\nabla\psi_s(x)$ exists and $\nabla\psi_s(x) = (1-s)x + s\nabla\psi(x)$, $s \in [0,1]$; similarly $\nabla^2\psi_s(x) = (1-s)I + s\nabla^2\psi(x)$, $s \in [0,1]$.

By Theorem A3.3 it turns out that ψ_s is the optimal transport map between μ and $\psi_{s\#}\mu$. But it is not needed here, what is used is convexity in s of the functions

$$G(s,x) := \big(\det \nabla^2\psi_s(x)\big)^{1-m}, \quad s \in [0,1],\ x \in \mathbb{R}^d \setminus N.$$

Forgetting the variable $x \in \mathbb{R}^d \setminus N$ we write

$$G(s) := \big(\det((1-s)I + sA)\big)^{1-m},$$

with $A = \nabla^2\psi(x)$ symmetric positive definite. If B is an orthogonal matrix for which $A = B^T D B$ where

$$D = \begin{bmatrix} d_1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & d_d \end{bmatrix}$$

with $d_i > 0$, $1 \le i \le d$, we obtain

$$G(s) = \left(\prod_{k=1}^d [(1-s) + sd_k]\right)^{1-m} > 0$$

and

$$\frac{d}{ds}\log G(s) = \frac{d}{ds}\sum_{k=1}^d \log((1-s)+sd_k)^{1-m} = \sum_{k=1}^d (1-m)\,\frac{d_k - 1}{1-s+sd_k}\,.$$

So

$$\frac{d^2}{ds^2}\log G(s) = -(1-m)\sum_{k=1}^d \frac{(d_k-1)^2}{((1-s)+sd_k)^2} > 0.$$

Therefore G is log-convex, hence G is convex and $G(1) - G(0) \ge G'(0^+)$. We have

$$G'(s) = G(s)\sum_{k=1}^d (1-m)\,\frac{d_k-1}{1-s+sd_k}\,,$$

hence

$$G'(0) = (1-m)\sum_{k=1}^{d}(d_k - 1) = (1-m)\operatorname{Tr} A$$
$$= (1-m)\operatorname{Tr}(\nabla^2\psi(x)) = -(m-1)\operatorname{Tr}(\nabla^2\psi(x) - I).$$

Therefore

$$\int_{\mathbb{R}^d} \widehat{\rho}(x)^m\,dx - \int_{\mathbb{R}^d} \rho(x)^m\,dx = \int_{\mathbb{R}^d\setminus N} \rho(x)^m[G(1,x) - G(0,x)]\,dx$$
$$= -(m-1)\int_{\mathbb{R}^d\setminus N} \rho(x)^m \operatorname{Tr}(\nabla^2\psi(x) - I)\,dx.$$

Notice that $x \mapsto \rho(x)^m \in W^{1,1}(\mathbb{R}^d)$ and is nonnegative, we can apply Theorem A3.4 of Appendix 3 to show that

$$-(m-1)\int_{\mathbb{R}^d} \rho^m \operatorname{Tr}\nabla^2\psi(x)\,dx \geq (m-1)\int_{\mathbb{R}^d} \langle\nabla\rho^m, \nabla\psi(x)\rangle\,dx. \tag{3.16}$$

Clearly

$$\int_{\mathbb{R}^d} \rho(x)^m \operatorname{Tr} I\,dx = \int_{\mathbb{R}^d} \rho(x)^m \operatorname{div}\operatorname{id}(x)\,dx = -\int_{\mathbb{R}^d} \langle\nabla\rho(x)^m, x\rangle\,dx.$$

Therefore

$$\int_{\mathbb{R}^d} \widehat{\rho}(x)^m\,dx - \int_{\mathbb{R}^d} \rho(x)^m\,dx = m(m-1)\int_{\mathbb{R}^d} \rho(x)^{m-1}\left\langle\nabla\rho(x), r_\mu^\sigma(x) - x\right\rangle\,dx. \qquad \square$$

3.3 Dénoument (Lösung)

In this last section we define the negative Gibbs–Boltzmann entropy functional ϕ_E on the space $(\mathcal{P}_2(\mathbb{R}^d), W_2(\cdot,\cdot))$, prove its lower semicontinuity and finally show that μ_t defined in (1.5) satisfies (EVI) on $(0,\infty)$ with $X = \mathcal{P}_2(\mathbb{R}^d)$, $d = W_2(\cdot,\cdot)$, $\alpha = 0$ and $\phi = \phi_E$. We first define

$$F(s) := \begin{cases} s\log s, & s > 0, \\ 0, & s = 0. \end{cases} \tag{3.17}$$

Then $F : [0,\infty) \to \mathbb{R}$ is continuous, strictly negative on $(0,1)$, strictly positive and increasing on $(1,\infty)$, strictly convex, and satisfies $\lim_{s\to 0} sF'(s) = 0$, $\lim_{s\to 0} F(s)/s^\alpha = 0$ for every $\alpha \in (0,1)$, $\lim_{s\to\infty} F(s)/s = \infty$.

Let F^- denote the negative part of F, i.e.

$$F^-(s) = \begin{cases} -F(s) & s \in [0,1), \\ 0 & s \in [1,\infty). \end{cases}$$

Then for every $\alpha \in (0,1)$ there exists $C_\alpha > 0$ ($C_\alpha = (\alpha e)^{-1}$) such that $F^-(s) \leq C_\alpha s^\alpha$, $s \geq 0$. As a consequence, if $\mu \in \mathcal{P}_2^a(\mathbb{R}^d)$ with density ρ, then

$$\int_{\mathbb{R}^d} F^-(\rho(s))\,ds < \infty. \tag{3.18}$$

Clearly $F^- \circ \rho$ is Borel measurable and nonnnegative. Therefore for any $\alpha \in (0, 1)$

$$\begin{aligned}\int_{\mathbb{R}^d} F^-(\rho(x))\,dx \le C_\alpha \int_{\mathbb{R}^d} \rho(x)^\alpha\,dx &= C_\alpha \int_{\mathbb{R}^d} \rho(x)^\alpha (1 + |x|_2)^{2\alpha} (1 + |x|_2)^{-2\alpha}\,dx \\ &\le C_\alpha \Big(\int_{\mathbb{R}^d} \rho(x)(1 + |x|_2)^2\,dx\Big)^\alpha \Big(\int_{\mathbb{R}^d} (1 + |x|_2)^{-2\alpha/(1-\alpha)}\,dx\Big)^{1-\alpha}.\end{aligned}$$

The first integral is finite since $\mu \in \mathcal{P}_2(\mathbb{R}^d)$ and the second is finite provided $\frac{2\alpha}{1-\alpha} > d$, i.e. $1 > \alpha > \frac{d}{d+2}$.

Now we can define the functional ϕ_E.

Definition 3.1. Let $F : [0, \infty) \to \mathbb{R}$ be as in (3.17). Then for $\mu \in \mathcal{P}_2(\mathbb{R}^d)$ the functional $\phi_E : \mathcal{P}_2(\mathbb{R}^d) \to (-\infty, +\infty]$ is defined as follows:

$$\phi_E(\mu) := \begin{cases} +\infty & \text{if } \mu \notin \mathcal{P}_2^a(\mathbb{R}^d), \\ +\infty & \text{if } \mu \in \mathcal{P}^a(\mathbb{R}^d) \text{ with } \mu = \rho \cdot \lambda_d \\ & \qquad \text{and } \int_{\mathbb{R}^d} F^+(\rho(x))\,dx = +\infty, \\ \int_{\mathbb{R}^d} F(\rho(x))\,dx & \text{if } \int_{\mathbb{R}^d} |F(\rho(x))|\,dx < \infty. \end{cases} \tag{3.19}$$

Lemma 3.2.

(i) $\phi_E(N(0, tI)) = -\frac{d}{2}[1 + \log(2\pi t)]$, $t > 0$.

(ii) *Let* $\mu \in \mathcal{P}_2^a(\mathbb{R}^d)$ *with density* $\rho \in L^\infty(\mathbb{R}^d)$. *Then* $\phi_E(\mu) < \infty$. *In particular, if* μ_t *is as in* (1.6) *with* $\mu_0 \in \mathcal{P}_2(\mathbb{R}^d)$ *and* $t > 0$ *then* $\phi_E(\mu_t) < \infty$. *Similarly, if* $\sigma \in \mathcal{P}_2^a(\mathbb{R}^d)$ *with* $\sigma = \widetilde{\rho} \cdot \lambda_d$, *and* $\widetilde{\rho} \in C(\mathbb{R}^d)$ *with compact support, then* $\phi_E(\sigma) < \infty$.

Proof. (i)

$$\int_{\mathbb{R}^d} \rho_t(x) \log \rho_t(x)\,dx = \int_{\mathbb{R}^d} \rho_t(x) \Big[\log \frac{1}{(2\pi t)^{d/2}} - \frac{1}{2t}|x|_2^2\Big]\,dx = -\frac{d}{2}[1 + \log(2\pi t)],$$

so

$$\phi_E(N(0, (2\pi e)^{-1}I)) = 0, \quad \phi_E(N(0, (2\pi)^{-1}I)) = -\frac{d}{2}.$$

(ii)

$$\int_{\mathbb{R}^d} F^+(\rho(x))\,dx = \int_{\{\rho(x)\ge 1\}} (\log \rho(x))\,\rho(x)\,dx \le \max\{0, \log \|\rho\|_{L^\infty(\mathbb{R}^d)}\}$$

and $\int_{\mathbb{R}^d} F^-(\rho_t(x))\,dx < \infty$ by (3.18), hence $\phi_E(\mu_t) < \infty$. □

Next we want to establish (3.1) when $\sigma \in \mathcal{P}_2^a(\mathbb{R}^d)$ has compact support. The idea is to approximate $F(s)$ by $F_m(s) := \frac{1}{m-1}(s^m - s)$. We shall use the following facts. Let $u > 0$, $1 \ge h > 0$.

If $u \ge 1$, then

$$(u - 1)u \ge \frac{1}{h}(u^{1+h} - u) \downarrow u \log u > 0 \quad \text{as } h \downarrow 0. \tag{3.20}$$

If $1 > u > 0$, then

$$\frac{1}{h}\,|u^{1+h} - u| \uparrow u\,|\log u| \quad \text{as } h \downarrow 0. \tag{3.21}$$

Let $\mu \in \mathcal{P}_2^a(\mathbb{R}^d)$ with density $\rho \in L^\infty(\mathbb{R}^d)$. We have $\int_{\mathbb{R}^d} |F(\rho(x))|\,dx < \infty$ by Lemma 3.2.

We claim

$$\int_{\mathbb{R}^d} F(\rho(x))\,dx = \lim_{m\downarrow 1} \int_{\mathbb{R}^d} \frac{1}{m-1}(\rho(x)^m - \rho(x))\,dx. \tag{3.22}$$

Indeed,

$$\begin{aligned}
&\int_{\mathbb{R}^d} F(\rho(x))\,dx = \int_{\{\rho(x)>1\}} F^+(\rho(x))\,dx - \int_{\{\rho(x)<1\}} F^-(\rho(x))\,dx \\
&= \int_{\{\rho(x)>1\}} \inf_{m>1}\Big\{\frac{1}{m-1}(\rho(x)^m - \rho(x))\Big\}dx - \int_{\{\rho(x)<1\}} \inf_{m>1}\Big\{\frac{1}{m-1}(\rho(x)^m - \rho(x))\Big\}dx.
\end{aligned}$$

Making use of (3.20) and (3.21) we obtain by Lebesgue monotone convergence

$$\int_{\{\rho(x)>1\}} \inf_{m>1}\Big\{\frac{1}{m-1}(\rho(x)^m - \rho(x))\Big\}dx = \lim_{m \underset{>}{\to} 1} \int_{\{\rho(x)>1\}} \frac{1}{m-1}(\rho(x)^m - \rho(x))\,dx$$

as well as

$$\int_{\{\rho(x)<1\}} \inf\Big\{\frac{1}{m-1}(\rho(x)^m - \rho(x))\Big\}dx = \lim_{m \underset{>}{\to} 1} \int_{\{\rho(x)<1\}} \frac{1}{m-1}(\rho(x)^m - \rho(x))\,dx.$$

This implies the claim.

Now, let $\widetilde{\rho}$ be the density of $\sigma \in \mathcal{P}_2^a(\mathbb{R}^d)$ with compact support. Combining (3.22) with (3.13) and using $\int_{\mathbb{R}^d} \widetilde{\rho}^2(x)\,dx = \int_{\mathbb{R}^d} \rho_t(x)\,dx = 1$, we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^d} F(\widetilde{\rho}(x))\,dx - \int_{\mathbb{R}^d} F(\rho_t(x))\,dx \\
&\qquad = \lim_{m\downarrow 1} \int_{\mathbb{R}^d} \frac{1}{m-1}\big\{(\widetilde{\rho}(x)^m - \widetilde{\rho}(x)) - (\rho_t(x)^m - \rho_t(x))\big\}\,dx \\
&\qquad\qquad = \lim_{m\downarrow 1} \int_{\mathbb{R}^d} \frac{1}{m-1}\big(\widetilde{\rho}(x)^m - \rho_t(x)^m\big)\,dx \\
&\quad \geq \varliminf_{m\downarrow 1} \int_{\mathbb{R}^d} m\rho(x)^{m-1}\left\langle \nabla\rho_t(x), r_\mu^\sigma(x) - x\right\rangle\,dx = \int_{\mathbb{R}^d} \left\langle \nabla\rho_t(x), r_\mu^\sigma(x) - x\right\rangle\,dx.
\end{aligned}$$

In the last equality we used the Lebesgue dominated convergence noticing that $1 < m \leq 2$, $\rho_t(x)^{m-1}\big|\big\langle \nabla\rho_t(x), r_\mu^\sigma(x) - x\big\rangle\big|$, $x \in \mathbb{R}^d$, where $\rho_t(x)^{m-1} \leq C = C(t,d)$ and

$$\begin{aligned}
\int_{\mathbb{R}^d} \big|\big\langle \nabla\rho_t(x), r_\mu^\sigma(x) - x\big\rangle\big|\,dx &= \int_{\mathbb{R}^d} \left|\left\langle \frac{\nabla\rho_t(x)}{\rho_t(x)}, r_\mu^\sigma(x) - x\right\rangle\right|\,d\mu_t(x) \\
&\leq \left\|\frac{\nabla\rho_t}{\rho_t}\right\|_{L^2(\mu_t;\mathbb{R}^d)} \big\|r_\mu^\sigma - \mathrm{id}\big\|_{L^2(\mu_t;\mathbb{R}^d)} < \infty.
\end{aligned}$$

Therefore (3.1) holds for any $\sigma \in \mathcal{P}_2^a(\mathbb{R}^d)$ with compact support, with $\phi = \phi_E$.

Combining what precedes with (2.21) and (2.4), we obtain the following result.

For every $\sigma \in \mathcal{P}_2^a(\mathbb{R}^d)$ with compact support and for every $t > 0$ such that $|\dot{\mu}|(t)$ exists as well as the derivative of $t \mapsto W_2^2(\mu_t, \sigma)$, we have

$$\frac{d}{dt}\frac{1}{2}W_2^2(\mu_t, \sigma) \leq \frac{\beta}{2}\left(\phi_E(\sigma) - \phi_E(\mu_t)\right) \tag{3.23}$$

where μ_t is defined in (1.5).

In the next step we show that the function $(0,\infty) \ni t \mapsto \phi_E(\mu_t)$ is nonincreasing. In view of semigroup property and the fact that $\phi_E(\mu_t) < \infty$ for $t > 0$ it is sufficient to prove that

$$\phi_E(S_\beta(t)\mu_0) \le \phi_E(\mu_0) \quad \text{for } t > 0 \tag{3.24}$$

whenever $\mu_0 \in \mathcal{P}_2^a(\mathbb{R}^d)$ and ρ_0, the density of μ_0 being continuous, positive and bounded. Notice that

$$\rho_t(x) = \int_{\mathbb{R}^d} p_{\beta t}(x-y)\rho_0(y)\, dy, \quad x \in \mathbb{R}^d,$$

hence by Young's inequality for convolutions we have

$$\Big(\int_{\mathbb{R}^d} (\rho_t(x))^m \, dx\Big)^{1/m} \le \Big(\int_{\mathbb{R}^d} (\rho_0(x))^m \, dx\Big)^{1/m}, \quad m > 1.$$

Therefore by (3.22), taking into account $\int_{\mathbb{R}^d} \rho_t(x)\, dx = \int_{\mathbb{R}^d} \rho_0(x)\, dx$, we obtain

$$\begin{aligned}\int_{\mathbb{R}^d} F(\rho_t(x))\, dx &= \lim_{m\downarrow 1} \frac{1}{m-1} \int_{\mathbb{R}^d} (\rho_t(x)^m - \rho_t(x))\, dx \\ &\le \lim_{m\downarrow 1} \frac{1}{m-1} \int_{\mathbb{R}^d} (\rho_0(x)^m - \rho_0(x))\, dx = \int_{\mathbb{R}^d} F(\rho_0(x))\, dx.\end{aligned}$$

This establishes (3.24).

Using the nonincreasingness of $t \mapsto \phi_E(\mu_t)$ and integrating (3.23) over $0 < t_1 < t_2$ we get

$$\frac{1}{2} W_2^2(\mu_{t_2}, \sigma) - \frac{1}{2} W_2^2(\mu_{t_1}, \sigma) \le (t_2 - t_1)\, \frac{\beta}{2}\, (\phi_E(\sigma) - \phi_E(\mu_{t_2})) \tag{3.25}$$

for every $0 < t_1 < t_2$, $\sigma \in \mathcal{P}_2^a(\mathbb{R}^d)$ with compact support. Observe that we have used the absolute continuity of $t \mapsto W_2^2(\mu_t, \sigma)$. In the "almost" final step we want to use in (3.25) sequences σ^n approximating $\sigma \in \mathcal{P}_2^a(\mathbb{R}^d)$ such that the density of σ denoted by $\widetilde{\rho}$ is essentially bounded above.

We define for n large enough:

$$\widetilde{\rho}_n(x) := \Big(\int_{\mathbb{R}^d} \widetilde{\rho}(x)\eta(\tfrac{x}{n})\, dx\Big)^{-1} \widetilde{\rho}_s(x)\eta(\tfrac{x}{n}), \quad x \in \mathbb{R}^d,$$

where $\eta \in C_c^\infty(\mathbb{R}^d)$, $0 \le \eta(x) \le 1$, $x \in \mathbb{R}^d$, and $\eta(x) = 1$, $|x|_2 \le 1$. Clearly $0 \le \widetilde{\rho}_n$, $\int_{\mathbb{R}^d} \widetilde{\rho}_n(x)\, dx = 1$ and $\widetilde{\rho}_n$ has compact support. Set $\sigma^n := \widetilde{\rho}_n \cdot \lambda_d$. Then $\sigma^n \in \mathcal{P}_2^a(\mathbb{R}^d)$. Moreover, by using the Lebesgue dominated convergence theorem we have

$$\lim_{n\to\infty} \int_{\mathbb{R}^d} \widetilde{\rho}(x)\eta(\tfrac{x}{n})\, dx = 1,$$

$$\lim_{n\to\infty} \int_{\mathbb{R}^d} g(x)\widetilde{\rho}_n(x)\, dx = \int_{\mathbb{R}^d} g(x)\widetilde{\rho}(x)\, dx$$

for every $g \in \mathrm{BC}(\mathbb{R}^d)$ and

$$\lim_{n\to\infty} \int_{\mathbb{R}^d} |x|_2^2 \widetilde{\rho}_n(x)\, dx = \int_{\mathbb{R}^d} |x|_2^2 \widetilde{\rho}(x)\, dx.$$

Therefore by Proposition 1.3(iii), $\lim\limits_{n\to\infty} W_2(\sigma^n,\sigma)=0$.

It remains to show that $\phi_E(\sigma^n)\to\phi_E(\sigma)$ as $n\to\infty$. Clearly

$$F(\widetilde{\rho}_n(x)) \overset{n\to\infty}{\longrightarrow} F(\widetilde{\rho}(x)) \quad \text{for a.a. } x\in\mathbb{R}^d.$$

Moreover, there exist $M_1, M_2>0$ such that

$$0\le \widetilde{\rho}_n(x)\le M_1\rho(x)\le M_2 \quad \text{for every } x\in\mathbb{R}^d,\ n\ge 1.$$

Therefore there exists $\overline{M}>0$ such that

$$F^+(\widetilde{\rho}_n(x)) = (\log\widetilde{\rho}_n(x))^+\widetilde{\rho}_n(x)\le \overline{M}\widetilde{\rho}(x) \quad \text{a.e. in } \mathbb{R}^d,$$

hence

$$\lim_{n\to\infty}\int_{\mathbb{R}^d} F^+(\widetilde{\rho}_n(x))\,dx = \int_{\mathbb{R}^d} F^+(\widetilde{\rho}(x))\,dx.$$

On the other hand we have seen that there exist $\alpha\in(\frac{d}{d+2},1)$ and $C_\alpha>0$ such that $F^-(s)\le C_\alpha s^\alpha$, $s\ge 0$. Therefore $F^-(\widetilde{\rho}_n(x))\le C_\alpha M_1^\alpha\widetilde{\rho}^\alpha(x)$, $x\in\mathbb{R}^d$, and $\int_{\mathbb{R}^d}\widetilde{\rho}^\alpha(x)\,dx<\infty$ by (3.18).

Consequently

$$\lim_{n\to\infty}\int_{\mathbb{R}^d} F^-(\widetilde{\rho}_n(x))\,dx = \int_{\mathbb{R}^d} F^-(\widetilde{\rho}(x))\,dx.$$

Therefore we have shown that $\phi_E(\sigma^n)\to\phi_E(\sigma)$.

Using the continuity of the metric, we can pass to the limit in (3.25) where σ is replaced by σ^n. We obtain

$$\frac{1}{2}W_2^2(\mu_{t_2},\sigma)-\frac{1}{2}W_2^2(\mu_{t_1},\sigma)\le (t_2-t_1)\,\frac{\beta}{2}\,(\phi_E(\sigma)-\phi_E(\mu_{t_2})) \tag{3.26}$$

for every $0<t_1<t_2$ and every $\sigma=\widetilde{\rho}\cdot\lambda_d$ with $\sigma\in\mathcal{P}_2^a(\mathbb{R}^d)$ and $\widetilde{\rho}$ essentially bounded above.

Finally we want to remove the boundedness condition on the density of σ. Let $\sigma\in\mathcal{P}_2^a(\mathbb{R}^d)$ and let ρ be its density. We consider the following approximations of ρ. Let $m\ge 1$. Set

$$z_m := \int_{\mathbb{R}^d}(\rho(x)\wedge m)\,dx. \tag{3.27}$$

Clearly $0<z_m\uparrow 1$ by the Lebesgue dominated convergence theorem.

$$\widetilde{\rho}_m(x) := z_m^{-1}(\rho(x)\wedge m), \quad x\in\mathbb{R}^d. \tag{3.28}$$

Then $\int_{\mathbb{R}^d}\widetilde{\rho}_m(x)\,dx=1$ and we set

$$\sigma_m := \widetilde{\rho}_m\cdot\lambda_d. \tag{3.29}$$

Clearly $\sigma_m\in\mathcal{P}_2^a(\mathbb{R}^d)$ and (3.26) holds with σ replaced by σ_m. We claim

$$\lim_{m\to\infty} W_2(\sigma_m,\sigma)=0, \tag{3.30}$$

$$\lim_{m\to\infty}\int_{\mathbb{R}^d} F^+(\widetilde{\rho}_m(x))\,dx = \int_{\mathbb{R}^d} F^+(\rho(x))\,dx, \tag{3.31}$$

$$\lim_{m\to\infty}\int_{\mathbb{R}^d} F^-(\widetilde{\rho}_m(x))\,dx = \int_{\mathbb{R}^d} F^-(\rho(x))\,dx, \tag{3.32}$$

As a consequence $\phi_E(\sigma_m)\overset{m\to\infty}{\longrightarrow}\phi_E(\sigma)$ and (3.26) holds for every $\sigma\in\mathcal{P}_2^a(\mathbb{R}^d)$. When $\sigma\in\mathcal{P}_2(\mathbb{R}^d)\setminus\mathcal{P}_2^a(\mathbb{R}^d)$, $\phi_E(\sigma)=+\infty$, hence (3.26) also holds.

Proof. (3.30). By Proposition 1.3(iii) it is sufficient to show that $\sigma_m \Rightarrow \sigma$ and $m_2(\sigma_m) \to m_2(\sigma)$ as $m \to \infty$. This is a direct consequence of the Lebesgue monotone convergence theorem and the fact that $z_m^{-1} \to 1$ as $m \to \infty$.

(3.31). Clearly $F^+(z_m^{-1}(\rho(x) \wedge m)) \to F^+(\rho(x))$ a.e. in $\mathbb{R}^d$ as $m \to \infty$. If $\int_{\mathbb{R}^d} F^+(\rho(x))\,dx < +\infty$ then $F^+(z_m^{-1}(\rho(x) \wedge m)) \le z_1^{-1} F^+(\rho(x))$ and the result follows from the Lebesgue dominated convergence theorem. If $\int_{\mathbb{R}^d} F^+(\rho(x))\,dx = +\infty$ then

$$z_m \cdot F^+(z_m^{-1}(\rho(x) \wedge m)) \ge \big((\log z_1^{-1}) + \log(\rho(x) \wedge m)\big)^+ (\rho(x) \wedge m)$$
$$\uparrow \big((\log z_1^{-1}) + \log \rho(x)\big)^+ \rho(x), \quad \text{as } m \uparrow \infty, \text{ a.e. in } \mathbb{R}^d.$$

Therefore

$$\varliminf_{m \to \infty} \int_{\mathbb{R}^d} F^+(z_m^{-1}(\rho(x) \wedge m))\,dx \ge \int_{\mathbb{R}^d} \big((\log z_1^{-1}) + \log \rho(x)\big)^+ \rho(x)\,dx = +\infty.$$

(3.32). Clearly $F^-(z_m^{-1}(\rho(x) \wedge m)) \to F^-(\rho(x))$ a.e. in $\mathbb{R}^d$ as $m \to \infty$. By (3.18) $\int_{\mathbb{R}^d} F^-(\rho(x))\,dx < \infty$. Moreover, as for (3.18), there exist $\alpha \in (\frac{d}{d+2}, 1)$, $C_\alpha > 0$ such that

$$F^-(z_m^{-1}(\rho(x) \wedge m)) \le C_\alpha z_m^{-\alpha} (\rho(x) \wedge m)^\alpha \le C_\alpha z_1^{-\alpha} \rho(x)^\alpha.$$

Since $\int_{\mathbb{R}^d} \rho(x)^\alpha\,dx < \infty$, we are done. $\square$

We conclude this section by proving the lower semicontinuity of ϕ_E.

Theorem 3.1. *Let* $\phi_E : \big(\mathcal{P}_2(\mathbb{R}^d), W_2(\cdot,\cdot)\big) \to (-\infty, +\infty]$ *be as in* (3.19). *Then*

(i) *The functional* ϕ_E *is lower semicontinuous.*

(ii) *The function* $(0,\infty) \ni t \mapsto \mu_t \in \mathcal{P}_2^a(\mathbb{R}^d)$ *defined in* (1.5) *satisfies* (EVI) *with* $\alpha = 0$ *and* $\phi := \frac{\beta}{2}\phi_E$. *Moreover,* $\lim_{t \to 0} W_2(\mu_t, \mu_0) = 0$.

Proof. (i) l.s.c. of ϕ_E.

Let $\mu, \mu^n \in \mathcal{P}_2(\mathbb{R}^d)$, $n \ge 1$, and $c \in \mathbb{R}$ be such that $\lim_{n \to \infty} W_2(\mu^n, \mu) = 0$ and $\phi_E(\mu^n) \le c$, $n \ge 1$. We have to show that $\phi_E(\mu) \le c$. Clearly $\mu^n \in \mathcal{P}_2^a(\mathbb{R}^d)$, $n \ge 1$, and let ρ_n denote the density of μ^n. As in the proof of (3.18), we have for some $\alpha \in (\frac{d}{d+2}, 1)$ and $C_\alpha > 0$,

$$\int_{\mathbb{R}^d} F^-(\rho_n(x))\,dx \le C_a \Big(\int_{\mathbb{R}^d} \rho_n(x)(1+|x|_2)^2\,dx\Big)^\alpha \Big(\int_{\mathbb{R}^d} (1+|x|_2)^{-2\alpha/(1-\alpha)}\,dx\Big)^{1-\alpha} \le M_1,$$

for some M_1 independent of n. Therefore

$$\int_{\mathbb{R}^d} F^+(\rho_n(x))\,dx \le c + M_1, \quad n \ge 1. \tag{3.33}$$

Since $F^+\colon [0,+\infty) \to [0,+\infty)$ is convex, continuous and satisfies $F^+(\rho)/\rho$ diverges as $\rho \to \infty$, it follows from Lemma 3.4 and Corollary 3.5 of [17] that $\mu \in \mathcal{P}_2^a(\mathbb{R}^d)$. If ρ denotes its density, we have $\int_{\mathbb{R}^d} F^+(\rho(x))\,dx < \infty$ and

$$\phi_E(\mu) = \int_{\mathbb{R}^d} F(\rho(x))\,dx < \infty.$$

It remains to show that $\int_{\mathbb{R}^d} F(\rho(x))\,dx \le c$. We have for any $R > 0$

(3.34)
$$\int_{\mathbb{R}^d\setminus B_R} F^-(\rho_n(x))\,dx \le C_a\Big(\int_{\mathbb{R}^d} \rho_n(x)(1+|x|_2)^2\,dx\Big)^{\alpha}\Big(\int_{\mathbb{R}^d\setminus B_R} (1+|x|_2)^{-2\alpha/(1-\alpha)}\,dx\Big)^{1-\alpha},$$

where $\alpha \in (\frac{d}{d+2}, 1)$. Since $W_2(\mu^n,\mu) \to 0$, the first integral in the RHS of (3.34) is bounded above by some constant C_1 independent of n. Therefore we obtain

$$\lim_{R\uparrow\infty} \sup_{n\ge 1} \int_{\mathbb{R}^d\setminus B_R} F^-(\rho_n(x))\,dx = 0, \tag{3.35}$$

where B_R denotes the ball of radius R centered at 0 in $\mathbb{R}^d$.

Since $[0,\infty) \ni s \mapsto F(s)$ is convex and $[0,\infty) \ni s \mapsto F^+(s)$ is nonnegative and convex, we have for any $0 < R < \infty$

$$\int_{B_R} F(\rho(x))\,dx \le \varliminf_{n\to\infty} \int_{B_R} F(\rho_n(x))\,dx, \tag{3.36}$$

$$\int_{\mathbb{R}^d\setminus B_R} F^+(\rho(x))\,dx \le \varliminf_{n\to\infty} \int_{\mathbb{R}^d\setminus B_R} F^+(\rho_n(x))\,dx. \tag{3.37}$$

Inequality (3.37) can be obtained as in Lemma 3.4 of [17] by proving first that

$$\int_{\mathbb{R}^d\setminus B_{R+\bar\varepsilon}} F^+(\rho(x))\,dx \le \varliminf_{n\to\infty} \int_{\mathbb{R}^d\setminus B_R} F^+(\rho^n(x))\,dx$$

for any $\varepsilon \in (0,R)$. This can be done by using $\varepsilon < \bar\varepsilon$ small enough in the proof. Then one lets $\bar\varepsilon$ tend to zero. In inequality (3.36) F is not nonnegative, but bounded below. So if $m := \min F$ we can add $\int_{B_R} m\,dx$, which is finite, to both sides of (3.36) in such way the new function $F(s) + m$ becomes nonnegative. Then we can establish (3.36) for this new function in the same way as in (3.37) proving it first for B_R replaced by $B_{R-\bar\varepsilon}$, $0 < \bar\varepsilon < R$ on the LHS. Therefore since for any $R < \infty$,

$$\int_{\mathbb{R}^d} F(\rho(x))\,dx \le \int_{B_R} F(\rho(x))\,dx + \int_{\mathbb{R}^d\setminus B_R} F^+(\rho(x))\,dx,$$

we obtain from (3.35), (3.36) and (3.37)

$$\begin{aligned}\int_{\mathbb{R}^d} F(\rho(x))\,dx &\le \varliminf_{n\to\infty} \int_{B_R} F(\rho_n(x))\,dx + \varliminf_{n\to\infty} \int_{\mathbb{R}^d\setminus B_R} F^+(\rho_n(x))\,dx \\ &\le \varliminf_{n\to\infty} \int_{B_R} F(\rho_n(x))\,dx + \varliminf_{n\to\infty} \int_{\mathbb{R}^d\setminus B_R} \big(F^+(\rho_n(x)) - F^-(\rho_n(x))\big)\,dx \\ &+ \sup_{k\ge 1} \int_{\mathbb{R}^d\setminus B_R} F^-(\rho_k(x))\,dx \le \varliminf_{n\to\infty} \int_{\mathbb{R}^d} F(\rho_n(x))\,dx + \sup_{k\ge 1} \int_{\mathbb{R}^d\setminus B_R} F^-(\rho_k(x))\,dx.\end{aligned}$$

By (3.35) letting R tend to $+\infty$ we obtain

$$\int_{\mathbb{R}^d} F(\rho(x))\,dx \le \varliminf_{n\to\infty} \int_{\mathbb{R}^d} F(\rho_n(x))\,dx \le c.$$

This completes the proof of the l.s.c. of ϕ_E.

(ii) The first part has been proved in Section 3.3 and the second part in Section 1.2. □

Appendix 1

The aim of this appendix is to recall, mostly without proofs, some results concerning functions of bounded variation.

Let (X, d) be a (not necessarily complete) metric space. Let $a, b \in \mathbb{R}$ with $a < b$ and let $u : [a, b] \to X$. Given a partition π, $a = t_0 < t_1 < \ldots < t_n = b$, let

$$V(\pi; u) := \sum_{i=1}^{n} d\big(u(t_{i-1}), u(t_i)\big).$$

Then u is said to be of *bounded variation* (with respect to the metric d) if $\sup_\pi V(\pi; u) < \infty$. We denote by $\mathrm{BV}([a, b]; X)$ the collection of all X-valued functions which are of bounded variation. We use the notation

$$V(u; [a, b]) := \sup_\pi V(\pi; u) \quad \text{over all partitions } \pi \text{ of } [a, b]. \tag{A1.1}$$

Clearly if $u \in \mathrm{Lip}([a, b]; X)$ then $u \in \mathrm{BV}([a, b]; X)$ and $V(u; [a, b]) \le [u]_{\mathrm{Lip}}(b-a)$. As in the case $X = \mathbb{R}$ one shows that if $u \in \mathrm{BV}([a, b]; X)$ and $c \in (a, b)$ then $u|_{[a,c]} \in \mathrm{BV}([a, c]; X)$, $u|_{[c,b]} \in \mathrm{BV}([c, b]; X)$ and

$$V(u; [a, b]) = V\big(u|_{[a,c]}; [a, c]\big) + V\big(u|_{[c,b]}; [c, b]\big). \tag{A1.2}$$

We shall denote by $V_u(t)$ the real-valued function defined by

$$V_u(t) := V(u; [a, t]), \quad t \in [a, b]. \tag{A1.3}$$

We have for $a \le s < t \le b$

$$d\big(u(s), u(t)\big) \le V_u(t) - V_u(s) = V(u; [s, t]). \tag{A1.4}$$

The function $V_u(\cdot)$ is nondecreasing and satisfies $V_u(a) = 0$.

Let $v : [a, b] \to X$. If there exists a nondecreasing function $M : [a, b] \to \mathbb{R}$ such that

$$d\big(v(s), v(t)\big) \le M(t) - M(s)$$

holds for all $a \le s < t \le b$, then $v \in \mathrm{BV}([a, b]; X)$ and $V_v(t) \le M(t) - M(a)$, $t \in [a, b]$.

It follows from (A1.4) that if $u \in \mathrm{BV}([a, b]; X)$, then the set where u is not continuous is at most countable. Also if $V_u(\cdot)$ is continuous then clearly u is continuous. On the other hand, it can be shown as in the case $X = \mathbb{R}$ that if u is right (resp. left) continuous at $t \in [a, b]$ then $V_u(\cdot)$ is also right (resp. left) continuous at t.

The next lemma is useful.

Lemma A1 ([7], Appendix)**.** *Let $u \in \mathrm{BV}([a, b]; X)$. Then we have for all h in $(0, b-a)$*

$$\int_a^{b-h} d\big(u(t), u(t+h)\big)\, dt \le hV(u; [a, b]). \tag{A1.5}$$

Proof. Since the set of discontinuity of u is at most countable, the same holds for the bounded functions $t \mapsto d(u(t), u(t+h))$, $t \mapsto V_u(t)$ and $t \mapsto V_u(t+h)$ on $[a, b-h]$. Hence these functions are integrable. Using (A1.4) we have

$$\begin{aligned} \int_a^{b-h} d\big(u(t), u(t+h)\big)\, dt &\le \int_a^{b-h} V_u(t+h) - V_u(t)\, dt \\ &= \int_{a+h}^{b} V_u(t)\, dt - \int_a^{b-h} V_u(t)\, dt \le \int_{b-h}^{b} V_u(t)\, dt \le hV_u(b) = hV(u; [a, b]). \quad \square \end{aligned}$$

A function $u \in C([a,b];X)$ is not necessarily of bounded variation but if u is absolutely continuous (see Definition 1.4), then it is of bounded variation and $V_u(\cdot) \in \mathrm{AC}[a,b]$ as in the case $X = \mathbb{R}$. Conversely, if $u \in \mathrm{BV}([a,b];X)$ and $V_u(\cdot) \in \mathrm{AC}[a,b]$ then $u \in \mathrm{AC}([a,b];X)$.

Let $v : [a,b] \to X$ be such that there exists a function $M : [a,b] \to X$ nondecreasing and absolutely continuous. Then by what precedes we have $v \in \mathrm{BV}([a,b];X)$ and $V_v(t) \le M(t) - M(a)$, $t \in [a,b]$. It is easy to verify that $V_v(\cdot) \in \mathrm{AC}[a,b]$ hence $v \in \mathrm{AC}([a,b];X)$. Notice that M is absolutely continuous iff there exists $m \in L^1(a,b)$ nonnegative such that $M(t) - M(s) = \int_s^t m(r)\,dr$, $a \le s < t \le b$. It follows that for $v : [a,b] \to X$ we have $v \in \mathrm{AC}([a,b];X)$ iff there exists $m \in L^1(a,b)$ nonnegative such that

$$\text{(A1.6)} \qquad d\big(v(s), v(t)\big) \le \int_s^t m(r)\,dr, \quad a \le s < t \le b.$$

In this case (A1.6) implies $V_v(t) - V_v(s) \le \int_s^t m(r)\,dr$, hence

$$\int_s^t \frac{d}{dr} V_v(r)\,dr \le \int_s^t m(r)\,dr, \quad a \le s < t \le b.$$

It follows that $\frac{d}{dr} V_v(r) \le m(r)$ a.e. in (a,b).

We conclude this Appendix by showing that if $u \in \mathrm{AC}([a,b];X)$, then the metric derivative $|\dot{u}|(t)$ (see Theorem 1.1) exists for almost all $t \in (a,b)$, $|\dot{u}| \in L^1(a,b)$ and

$$|\dot{u}|(t) = \frac{d}{dt} V_u(t) \quad \text{a.e. in } (a,b).$$

Proof ([2], Theorem 1.1.2). Let $u \in \mathrm{AC}([a,b];X)$ and let N_u be a subset of (a,b) with measur zero such that $\frac{d}{dt} V_u(t)$ exists for every $t \in (a,b) \setminus N_u$. Since $u([a,b])$ is compact in X, it is separable. There exists a countable subset E of $u([a,b])$ which is dense in $u([a,b])$. For every $e \in E$ the functions $d(e, u(\cdot)) \in \mathrm{AC}[a,b]$ and let N_e be a subset of (a,b) with measur zero such that $\frac{d}{dt} d(e, u(t))$ exists for every $t \in (a,b) \setminus N_e$.

Set $N := N_u \cup \bigcup\limits_{e \in E} N_e$. For $t \in (a,b) \setminus N$ set

$$\ell(t) := \sup_{e \in E} \Big| \frac{d}{dt}\, d(e, u(t)) \Big| \quad \text{and } \ell(t) = 0, \ t \in N.$$

Then ℓ is nonnegative and measurable. We have

$$d\big(u(s), u(t)\big) = \sup_{e \in E} \big| d(e, u(s)) - d(e, u(t)) \big| \le \int_s^t \ell(r)\,dr, \quad a \le s < t \le b.$$

Let $t \in (a,b) \setminus N$. Then

$$\begin{aligned} \ell(t) = \sup_{e \in E} \lim_{s \to t} \frac{\big| d(e, u(t)) - d(e, u(s)) \big|}{|t-s|} &\le \lim_{s \to t} \frac{\big| d(u(t), u(s)) \big|}{|t-s|} \\ &\le \lim_{s \to t} \frac{|V_u(t) - V_u(s)|}{|t-s|} = \frac{d}{dt} V_u(t). \end{aligned}$$

It follows that $\ell \in L^1(a,b)$. Let N_ℓ be a subset of (a,b) of measur zero such that every $t \in (a,b) \setminus N_\ell$ is a Lebesgue point of ℓ. For every $t \in (a,b) \setminus N_\ell$ we have

$$\varlimsup_{s \to t} \frac{d(u(s), u(t))}{|t-s|} \le \ell(t).$$

Hence for every $t \in (a,b) \setminus (N \cup N_\ell)$ we have

$$\overline{\lim_{s\to t}} \frac{d(u(s),u(t))}{|t-s|} \le \lim_{s\to t} \frac{d(u(s),u(t))}{|t-s|} \le \frac{d}{dt} V_u(t).$$

Therefore on this set the metric derivative $|\dot{u}|(t)$ exists and $|\dot{u}|(t) = \ell(t) \le \frac{d}{dt}V_u(t)$.

On the other hand, since $d(u(s),u(t)) \le \int_s^t \ell(r)\,dr$, $a \le t < s \le b$, we have $\frac{d}{dt}V_u(t) \le \ell(t)$ a.e. in (a,b). It follows that $|\dot{u}|(t) = \frac{d}{dt}V_u(t)$ a.e. in (a,b). $\square$

Appendix 2

In this appendix we recall a smooth version of the formula of change of variables in integrals. As a consequence we establish the absolute continuity of some image measures and give a formula for their densities. Finally as a simple application we prove inequality (1.33).

Lemma A2.1. *Let $f \in C^1(\mathbb{R}^d;\mathbb{R}^d)$ be bijective and satisfy $|Jf(x)| > 0$ for every $x \in \mathbb{R}^d$. Here $Jf(x)$ is the Jacobian (determinant) of f at x, i.e.*

$$Jf(x) := \det\left[\frac{\partial}{\partial x_j} f_i(x)\right]_{1\le i,j\le d}.$$

Then for every $g : \mathbb{R}^d \to [0,\infty]$ Borel measurable we have

$$\int_{\mathbb{R}^d} g(y)\,dy = \int_{\mathbb{R}^d} g(f(x))|Jf(x)|\,dx. \tag{A2.1}$$

Moreover if $\mu \in \mathcal{P}^a(\mathbb{R}^d)$ with density ρ then $f_\#\mu \in \mathcal{P}^a(\mathbb{R}^d)$ with density $\tilde{\rho}$ given by

$$\tilde{\rho}(y) := \frac{\rho \circ f^{-1}(y)}{|\det(Jf)(f^{-1}(y))|}, \quad y \in \mathbb{R}^d. \tag{A2.2}$$

We recall how (A2.2) can be deduced from (A2.1). Let $g \in C_c(\mathbb{R}^d)$ (continuous with compact support). Then clearly

$$\int_{\mathbb{R}^d} |g(y)|\,d(f_\#\mu)(y) < \infty$$

and we have

$$\int_{\mathbb{R}^d} g(y)\,d(f_\#\mu)(y) = \int_{\mathbb{R}^d} (g\circ f)(x)\rho(x)\,dx = \int_{\mathbb{R}^d} \frac{(g\circ f)(x)\rho(x)}{|\det(Jf)(x)|}\,|\det(Jf)(x)|\,dx.$$

This implies the claim.

As an example we consider $f(x) := \lambda x$ with $\lambda > 0$. Then

$$\widetilde{\rho}(y) = \lambda^{-d}\rho(\lambda^{-1}y), \quad y \in \mathbb{R}^d. \tag{A2.3}$$

Now let $\mu = N(0,I)$ with $\rho(x) := \frac{1}{(2\pi)^{d/2}} \mathrm{e}^{-|x|_2^2/2}$. Set $f_1(x) := \lambda_1 x$ and $f_2(x) := \lambda_2 x$, $\lambda_1, \lambda_2 > 0$, and $(f_1,f_2)(x) := (\lambda_1 x, \lambda_2 x)$, $x \in \mathbb{R}^d$. Then $(f_1,f_2)_\#\mu \in \Gamma(f_{1\#}\mu, f_{2\#}\mu)$.

Notice that $f_{1\#}\mu,\ f_{2\#}\mu \in \mathcal{P}_p(\mathbb{R}^d)$ for every $p \geq 1$. Therefore,

$$\begin{aligned} W_p^p(f_{1\#}\mu, f_{2\#}\mu) &\leq \int_{\mathbb{R}^d\times\mathbb{R}^d} |x_1 - x_2|_2^p \, d(f_1, f_2)_{\#}\mu(x_1, x_2) \\ &= \int_{\mathbb{R}^d} |\lambda_1 x - \lambda_2 x|_2^p \, d\mu(x) = |\lambda_1 - \lambda_2|^p \int_{\mathbb{R}^d} |x|_2^p \, d\mu(x) \\ &= m_p(N(0, I))|\lambda_1 - \lambda_2|^p = a_{p,d}|\lambda_1 - \lambda_2|^p. \end{aligned}$$

Choosing $\lambda_1 := (\beta t)^{1/2}$, $\lambda_2 := (\beta s)^{1/2}$ with $0 < s, t$ and using (A2.3) we obtain

$$W_p(N(0, \beta t I), N(0, \beta s I)) \leq a_{p,d}^{1/p} \beta^{1/2} |t^{1/2} - s^{1/2}|,$$

which together with (1.32) implies (1.33).

Appendix 3

The aim of this Appendix is to prove some results used in Section 3.

Lemma A3.1. *Let $\mu \in \mathcal{P}_2^a(\mathbb{R}^d)$ with density $\rho > 0$ λ_d-a.e. in $\mathbb{R}^d$. Let $\psi \in C^1(\mathbb{R}^d)$ be convex with $\nabla\psi \in L^2(\mu; \mathbb{R}^d)$. Then*

(i) $\sigma := \nabla\psi_{\#}\mu \in \mathcal{P}_2(\mathbb{R}^d)$,

(ii) r_μ^σ*, the optimal transport map introduced in Theorem* 2.1(ii)*, satisfies*

$$r_\mu^\sigma = \nabla\psi \quad \textit{in } L^2(\mu; \mathbb{R}^d). \tag{A3.1}$$

Proof.

(i) $\nabla\psi \in C(\mathbb{R}^d, \mathbb{R}^d)$, hence is Borel measurable, therefore $\nabla\psi_{\#}\mu \in \mathcal{P}(\mathbb{R}^d)$ is well defined. Moreover, we have

$$\int_{\mathbb{R}^d} |x|_2^2 d(\nabla\psi_{\#}\mu)(x) = \int_{\mathbb{R}^d} |\nabla\psi(x)|_2^2 \, d\mu(x) < \infty,$$

hence $\sigma := \nabla\psi_{\#}\mu \in \mathcal{P}_2(\mathbb{R}^d)$.

(ii) Let $\gamma := (\mathrm{id}, \nabla\psi)_{\#}\mu \in \Gamma(\mu, \sigma)$. For any open ball $B_r(z)$, $r > 0$, $z \in \mathbb{R}^d \times \mathbb{R}^d$, such that $B_r(z) \cap \{(x, \nabla\psi(x)) : x \in \mathbb{R}^d\} = \varnothing$, we have

$$\int_{B_r(z)} d\gamma(x, y) = \int_{\mathbb{R}^d\times\mathbb{R}^d} \mathbb{1}_{B_r(z)}(x, y) \, d\gamma(x, y) = \int_{\mathbb{R}^d} \mathbb{1}_{B_r(z)}(x, \nabla\psi(x)) \, d\mu(x) = 0.$$

Therefore $\operatorname{supp}\gamma \subseteq \{(x, \nabla\psi(x)) : x \in \mathbb{R}^d\}$ (see Definition 1.5).

Moreover, the set $\{(x, \nabla\psi(x)) : x \in \mathbb{R}^d\}$ is cyclically monotone in $\mathbb{R}^d \times \mathbb{R}^d$ since it is the subdifferential of ψ (see Appendix of [17]). Therefore it verifies condition (1.42) of Theorem 1.3, hence $\gamma \in \Gamma_0(\mu, \sigma)$. By Theorem 2.1(i) and (ii) we have $\nabla\psi = r_\mu^\sigma$ in $L^2(\mu; \mathbb{R}^d)$, hence also, since $\rho > 0$ λ_d-a.e., $\nabla\psi(x) = r_\mu^\sigma(x)$ λ_d-a.e. □

In Lemma A3.1 we can always take $\nabla\psi$ identically zero. Then $\sigma = \delta_0$ the Dirac measure at 0. One may ask under which conditions on $\nabla\psi$ as in Lemma A3.1 we have $\sigma \in \mathcal{P}_2^a(\mathbb{R}^d)$.

Assuming $\nabla\psi : \mathbb{R}^d \to \mathbb{R}^d$ is injective, a necessary and sufficient condition for $\nabla\psi_\# \mu$ to belong to $\mathcal{P}_2^a(\mathbb{R}^d)$ can be given in terms of the so-called Aleksandrov derivative (Hessian) of ψ. We recall its definition in the case of $\psi \in C^1(\mathbb{R}^d)$, ψ convex.

We say that the function ψ has an *Aleksandrov Hessian* at $x_0 \in \mathbb{R}^d$ if there exists a $d \times d$ matrix A such that

$$\text{(A3.2)}\quad \psi(x) - \psi(x_0) - \langle \nabla\psi(x_0), x - x_0\rangle - \frac{1}{2}(x - x_0)^T A(x - x_0) \\ = o(|x - x_0|_2^2) \quad \text{as } x \to x_0.$$

If such A exists, it is unique and symmetric nonnegative definite. It is denoted by $\nabla^2\psi(x_0)$. It turns out that such $\nabla^2\psi(x_0)$ exists for λ_d-almost all $x_0 \in \mathbb{R}^d$.

We are now in a position to formulate the promised necessary and sufficient condition.

Lemma A3.2. *Let $\mu \in \mathcal{P}_2^a(\mathbb{R}^d)$ with density $\rho > 0$ λ_d-a.e. in $\mathbb{R}^d$. Let $\psi \in C^1(\mathbb{R}^d)$ be convex with $\nabla\psi \in L^2(\mu; \mathbb{R}^d)$. Suppose moreover that there exists a Borel set $\Sigma \subset \mathbb{R}^d$ such that $\lambda_d(\mathbb{R}^d \setminus \Sigma) = 0$ and $\nabla\psi|_\Sigma : \Sigma \to \mathbb{R}^d$ is injective. Then $\nabla\psi_\#\mu \in \mathcal{P}_2^a(\mathbb{R}^d)$ iff*

$$\text{(A3.3)}\qquad \nabla^2\psi(x_0) \text{ is positive definite for } \lambda_d\text{-a.a. } x_0 \in \mathbb{R}^d.$$

If this is the case and $\nabla\psi_\#\mu = \widetilde{\rho} \cdot \lambda_d$ then

$$\text{(A3.4)}\qquad \widetilde{\rho}\big((\nabla\psi)^{-1}(x)\big) = \frac{\rho(x)}{\det \nabla^2\psi(x)}$$

for every $x \in \Sigma$ such that $\nabla^2\psi(x) > 0$.

Moreover, if $F : [0, \infty) \to [0, \infty]$, $F(0) = 0$, is a Borel function, then

$$\text{(A3.5)}\qquad \int_{\mathbb{R}^d} F(\widetilde{\rho}(y))\,dy = \int_{\mathbb{R}^d} F\Big(\frac{\rho(x)}{\det \nabla^2\psi(x)}\Big) \det \nabla^2\psi(x)\,dx.$$

In particular, if ψ is replaced by ψ_ε where $\varepsilon > 0$ and $\psi_\varepsilon(x) := \frac{\varepsilon}{2}|x|_2^2 + \psi(x)$, $x \in \mathbb{R}^d$, we can take $\Sigma = \mathbb{R}^d$, $\nabla\psi_\varepsilon$ is also surjective and condition (A3.3) is always satisfied. $\psi_{\varepsilon\,\#}\mu \in \mathcal{P}_2^a(\mathbb{R}^d)$ and (A3.4), (A3.5) hold with ψ replaced by ψ_ε. Moreover,

$$\text{(A3.6)}\qquad \lim_{\varepsilon\to 0} W_2(\nabla\psi_{\varepsilon\,\#}\mu, \nabla\psi_\#\mu) = 0.$$

Proof. The first part of Lemma A3.2 is a direct consequence of Lemma 5.5.3, formula (5.5.3) and Theorem 5.5.4 in [2]. Concerning the second part with $\varepsilon > 0$, it is well known that the map $\nabla\psi_\varepsilon : \mathbb{R}^d \to \mathbb{R}^d$ is bijective. Indeed, for every $y \in \mathbb{R}^d$ there exists one and only one $\overline{x} \in \mathbb{R}^d$ such that

$$\psi_\varepsilon(\overline{x}) - \langle y, \overline{x}\rangle \le \psi_\varepsilon(x) - \langle y, x\rangle \quad \text{for all } x \in \mathbb{R}^d,$$

and this $\overline{x} \in \mathbb{R}^d$ satisfies $\nabla\psi_\varepsilon(\overline{x}) = y$. Notice that condition (A3.3) is satisfied since $\nabla^2\psi_\varepsilon(x_0) = \varepsilon I + \nabla^2\psi(x_0)$ whenever $\nabla^2\psi(x_0)$ exists.

Finally,

$$W_2^2(\nabla\psi_{\varepsilon\,\#}\mu, \nabla\psi_\#\mu) \le \int_{\mathbb{R}^d\times\mathbb{R}^d} \big|\nabla\psi_\varepsilon(x) - \nabla\psi(x)\big|_2^2\,d\mu(x) = \varepsilon^2 m_2(\mu) \to 0 \quad \text{as } \varepsilon \to 0. \qquad \square$$

In what follows we shall consider situations where the function ψ is not necessarily C^1.

Definition A3.1. Let Ω be a nonempty open convex subset of $\mathbb{R}^d$ and let $\psi : \Omega \to \mathbb{R}$ be a convex function. Let $x_0 \in \Omega$ be such that ψ is differentiable at x_0. We shall denote the set of differentiability of ψ by $D(\nabla\psi)$. We shall say that ψ has an *Aleksandrov derivative* (or *Hessian*) at x_0 if

(i) $x_0 \in D(\nabla\psi)$ and

(ii) (A3.2) holds for some $d \times d$ matrix A.

If such a matrix exists it is unique and nonnegative definite. It is denoted by $\nabla^2\psi(x_0)$, and the set $\{x_0 \in \nabla\psi : \nabla^2\psi(x_0) \text{ exists}\}$ will be denoted by $D(\nabla^2\psi)$.

Theorem A3.1 (see [3], Theorem 1.4 (Aleksandrov))**.** *Let Ω be a nonempty open convex subset of $\mathbb{R}^d$ and let $\psi : \Omega \to \mathbb{R}$ be a convex function. Then*

(i) *ψ is locally Lipschitz continuous,*

(ii) *$D(\nabla\psi)$ is a Borel set and $\lambda_d(\Omega \setminus D(\nabla\psi)) = 0$,*

(iii) *$\lambda_d(\Omega \setminus D(\nabla^2\psi)) = 0$,*

(iv) *$\nabla\psi : D(\nabla\psi) \to \mathbb{R}^d$ is monotone and Borel.*

Next we consider a sharpening of Theorem 2.1(ii).

Theorem A3.2 (see [3], Theorem 2.3 ii), iii) and Theorem 4.4 of [17])**.** *Let $\mu \in \mathcal{P}_2^a(\mathbb{R}^d)$ and $\sigma \in \mathcal{P}_2(\mathbb{R}^d)$. Then*

(i) *there exists a convex function $\psi : \mathbb{R}^d \to (-\infty, +\infty]$, whose finiteness domain $D(\psi)$ has nonempty interior and satisfies*

$$\mu(\mathbb{R}^d \setminus D(\psi)) = \mu(\mathbb{R}^d \setminus D(\nabla\psi)) = 0, \tag{A3.7}$$

such that $\nabla\psi = r_\mu^\sigma$ in $L^2(\mu; \mathbb{R}^d)$.

(ii) *If $\sigma \in \mathcal{P}_2^a(\mathbb{R}^d)$ as well, then*

  (a) *$r_\sigma^\mu \circ r_\mu^\sigma = \mathrm{id}$ μ-a.e. in $\mathbb{R}^d$,*
  $r_\mu^\sigma \circ r_\sigma^\mu = \mathrm{id}$ σ-a.e. in $\mathbb{R}^d$,
  moreover the set $\{x \in D(\nabla\psi) : \nabla^2\psi(x) \text{ exists and } \nabla^2\psi(x) \text{ is not invertible}\}$ is μ-negligible.

  (b) *If $F : [0, \infty) \to [0, \infty]$ is Borel measurable and satisfies $F(0) = 0$, then* (A3.5) *holds.*

(iii) *If, in addition, ρ—the density of μ—satisfies $\rho > 0$ λ_d-a.e. in $\mathbb{R}^d$, then $D(\psi) = \mathbb{R}^d$, ψ is locally Lipschitz continuous, $\nabla\psi(x)$ exists λ_d-a.e. as well as $\nabla^2\psi(x)$, and $\nabla^2\psi(x)$ is invertible a.e. in $\mathbb{R}^d$.*

Finally, we give a sharpening of Lemmata A3.1 and A3.2.

Let $\mu \in \mathcal{P}_2(\mathbb{R}^d)$, $\psi : \mathbb{R}^d \to \mathbb{R}$ convex, be as in Lemma A3.1 but without the condition $\psi \in C^1(\mathbb{R}^d)$. Let $D(\nabla\psi)$ be the set of differentiability of ψ or equivalently the set $\{x \in \mathbb{R}^d : \partial\psi(x) \text{ is a singleton}\}$ (see Appendix of [17]). In view of Theorem A3.1, $D(\nabla\psi)$ is a Borel set and the map $\nabla\psi : D(\nabla\psi) \to \mathbb{R}^d$ is Borel. Therefore for A Borel in $\mathbb{R}^d$,

$(\nabla\psi)^{-1}(A)$ is Borel in $\mathbb{R}^d$ and its measure μ is well defined. It will be called the *push-forward* of μ through $\nabla\psi$ and $(\nabla\psi)_{\#}\mu(x) \in \mathcal{P}(\mathbb{R}^d)$. Moreover

$$\int_{\mathbb{R}^d} g(x)\, d(\nabla\psi)_{\#}\mu(x) = \int_{D(\nabla\psi)} g(\nabla\psi(x))\, d\mu(x)$$

for every $g : \mathbb{R}^d \to [0, \infty]$ Borel function.

It is shown in [17, Lemma 4.1] that

$$(\nabla\psi)_{\#}\mu(A) = \mu(\partial\psi^*(A)),$$

for every Borel set A in $\mathbb{R}^d$ where ψ^* is the conjugate of ψ.

We are now in position to state

Theorem A3.3. *Let $\mu \in \mathcal{P}_2^a(\mathbb{R}^d)$, $\psi : \mathbb{R}^d \to \mathbb{R}$ convex, be as in Lemma* A3.1*, but* without *the assumption $\psi \in C^1(\mathbb{R}^d)$. Then*

(i) *the conclusions of Lemma* A3.1 *hold;*

(ii) *if, in addition, there exists a Borel set $\Sigma \subset D(\nabla\psi)$ such that $\lambda_d(\mathbb{R}^d \setminus \Sigma) = 0$ and $\nabla\psi|_{\Sigma} : \Sigma \to \mathbb{R}^d$ is injective, then $(\nabla\psi)_{\#}\mu \in \mathcal{P}_2^a(\mathbb{R}^d)$ iff* (A3.3) *holds. In this case* (A3.4) *and* (A3.5) *hold;*

(iii) *if ψ is replaced by ψ_ε as in Lemma* A3.2*, we can take $\Sigma := D(\nabla\psi_\varepsilon) = D(\nabla\psi)$. Then condition* (A3.3) *is always satisfied, $\nabla\psi_{\varepsilon\,\#}\mu \in \mathcal{P}_2^a(\mathbb{R}^d)$ and* (A3.4)*,* (A3.5) *hold with ψ replaced by ψ_ε. Moreover,* (A3.6) *holds.*

Proof. The proof is quite similar to the proofs of Lemmata A3.1 and A3.2 and therefore it is omitted. ☐

We conclude the Appendix by stating a result due to Calderón–Zygmund (see [1]) which allows us to relate the trace of the Aleksandrov derivative of a convex function with the divergence in the sense of distribution of its gradient.

Theorem A3.4. *Let $\psi : \mathbb{R}^d \to \mathbb{R}$ be convex and Lipschitz continuous. Let $D(\nabla\psi)$, $D(\nabla^2\psi)$ be as in Definition* A3.1*. Then $\nabla\psi$ is bounded and $|\nabla\psi|_2 \le d^{1/2}[\psi]_{\mathrm{Lip}}$.*

Moreover, by the Calderón–Zygmund theorem we have

$$\int_{\mathbb{R}^d} u\, \mathrm{tr}(\nabla^2\psi)\, dx \le -\int_{\mathbb{R}^d} \langle \nabla u, \nabla\psi \rangle\, dx \tag{A3.8}$$

for every $u \in W^{1,1}(\mathbb{R}^d)$ which is nonnegative.

Appendix 4

Proposition A4.1. *Let $\mu \in \mathcal{P}_2^a(\mathbb{R}^d)$, $\sigma \in \mathcal{P}_2(\mathbb{R}^d)$ and $r_\mu^\sigma \in L^2(\mu; \mathbb{R}^d)$ be the unique optimal transport map introduced in Theorem* 2.1*. Then $r_\mu^\sigma \in \mathrm{Tan}_\mu(\mathcal{P}_2(\mathbb{R}^d))$, defined in Definition* 2.1*.*

In the proof we shall use the following

Lemma A4.1. *Let* $\mu \in \mathcal{P}_2^a(\mathbb{R}^d)$, $\sigma, \sigma^n \in \mathcal{P}_2(\mathbb{R}^d)$ *and let* r_μ^σ, $r_\mu^{\sigma^n}$ *be the corresponding optimal transport maps. Then*

$$\text{(A4.1)} \qquad \lim_{n\to\infty} W_2(\sigma^n, \sigma) \quad \textit{iff} \quad \lim_{n\to\infty} \left\| r_\mu^{\sigma^n} - r_\mu^\sigma \right\|_{L^2(\mu;\mathbb{R}^d)} = 0.$$

Proof. (if)

$$W_2^2(\sigma^n, \sigma) \leq \int_{\mathbb{R}^d} |r_\mu^{\sigma^n}(x) - r_\mu^\sigma(x)|_2^2 \, d\mu(x)$$

since $(r_\mu^{\sigma^n}, r_\mu^\sigma)_{\#}\mu \in \Gamma(\sigma^n, \sigma)$.

(only if)

First we show

$$\text{(A4.2)} \qquad \lim_{n\to\infty} \int_{\mathbb{R}^d} \left\langle r_\mu^{\sigma^n}(x), u(x) \right\rangle \, d\mu(x) = \int_{\mathbb{R}^d} \left\langle r_\mu^\sigma(x), u(x) \right\rangle \, d\mu$$

for every $u \in \mathrm{BC}(\mathbb{R}^d)$.

To this end we consider the sequence $\gamma_\mu^{\sigma^n}$ of optimal transport plans in $\Gamma_0(\mu, \sigma^n)$, $n \geq 1$, whose existence and uniqueness is guaranteed by Theorem 1.2(i). Since $W_2(\sigma^n, \sigma) \to 0$ as $n \to \infty$, we have $\sigma^n \Rightarrow \sigma$ and $m_2(\sigma^n) \to m_2(\sigma)$, $n \to \infty$, as a consequence of Proposition 1.3(iii). The uniqueness part in Theorem 1.2(i) together with Proposition 1.2(ii) implies that $\gamma_\mu^{\sigma^n} \Rightarrow \gamma_\mu^\sigma$ in $\mathcal{P}(\mathbb{R}^d \times \mathbb{R}^d)$. Since $m_2(\mu), m_2(\sigma), m_2(\sigma^n) < \infty$, $n \geq 1$, we have $\gamma_\mu^\sigma \in \mathcal{P}_2(\mathbb{R}^d \times \mathbb{R}^d)$, $\gamma_\mu^{\sigma^n} \in \mathcal{P}_2(\mathbb{R}^d \times \mathbb{R}^d)$. Moreover, since $m_2(\sigma^n) \to m_2(\sigma)$, it follows from Proposition 1.3(iii) (in $\mathbb{R}^d \times \mathbb{R}^d$) that

$$\lim_{n\to\infty} \int_{\mathbb{R}^d \times \mathbb{R}^d} f(x,y) \, d\gamma_\mu^{\sigma^n}(x,y) = \int_{\mathbb{R}^d \times \mathbb{R}^d} f(x,y) \, d\gamma_\mu^\sigma(x,y)$$

for every $f \in C(\mathbb{R}^d \times \mathbb{R}^d)$ such that there exist $C_1, C_2 > 0$ for which

$$|f(x,y)| \leq C_1 + C_2(|x|_2^2 + |y|_2^2), \quad x, y \in \mathbb{R}^d.$$

Choosing $f(x,y) := \langle u(x), y \rangle$, $x, y \in \mathbb{R}^d$, $u \in \mathrm{BC}(\mathbb{R}^d)$, we have $f \in C(\mathbb{R}^d \times \mathbb{R}^d)$ and

$$|f(x,y)| \leq \frac{1}{2} \|u\|_\infty^2 + \frac{1}{2} |y|_2^2, \quad x, y \in \mathbb{R}^d.$$

Therefore f satisfies the required conditions and

$$\begin{aligned}\int_{\mathbb{R}^d} \left\langle u(x), r_\mu^{\sigma^n}(x) \right\rangle \, d\mu(x) &= \int_{\mathbb{R}^d \times \mathbb{R}^d} \langle u(x), y \rangle \; d\gamma_\mu^{\sigma^n}(x,y) \\ &\stackrel{n\to\infty}{\longrightarrow} \int_{\mathbb{R}^d \times \mathbb{R}^d} \langle u(x), y \rangle \; d\gamma_\mu^\sigma(x,y) = \int_{\mathbb{R}^d} \left\langle u(x), r_\mu^\sigma(x) \right\rangle \, d\mu(x),\end{aligned}$$

since $\gamma_\mu^{\sigma^n} = \mathrm{id} \times r_\mu^{\sigma^n}$ and $\gamma_\mu^\sigma = \mathrm{id} \times r_\mu^\sigma$, by Theorem 1.2(ii).

This completes the proof of (A4.2). Next we prove that $r_\mu^{\sigma^n}$ converges weakly to r_μ^σ in $L^2(\mu; \mathbb{R}^d)$. This will be a consequence of the boundedness of $\left\| r_\mu^{\sigma^n} \right\|_{L^2(\mu;\mathbb{R}^d)}$, of (A4.2) with $u \in \mathrm{BC}(\mathbb{R}^d)$ and of the fact that $\mathrm{BC}(\mathbb{R}^d)$ is dense in $L^2(\mu; \mathbb{R}^d)$. Indeed, if $\left\| r_\mu^{\sigma^n} \right\|_{L^2(\mu;\mathbb{R}^d)} \leq M$, $n \geq 1$, for some $M > 0$, given $\overline{u} \in L^2(\mu; \mathbb{R}^d)$, we have

$$\begin{aligned}&\left| \int_{\mathbb{R}^d} \left\langle r_\mu^{\sigma^n}, \overline{u} \right\rangle \, d\mu - \int_{\mathbb{R}^d} \left\langle r_\mu^\sigma, \overline{u} \right\rangle \, d\mu \right| \\ &\qquad \leq \left(M + \left\| r_\mu^\sigma \right\|_{L^2(\mu;\mathbb{R}^d)} \right) \|\overline{u} - u\|_{L^2(\mu;\mathbb{R}^d)} + \left| \int_{\mathbb{R}^d} \left\langle r_\mu^{\sigma^n}, u \right\rangle \, d\mu - \int_{\mathbb{R}^d} \left\langle r_\mu^\sigma, u \right\rangle \, d\mu \right|.\end{aligned}$$

Given $\varepsilon > 0$, choosing $u \in \mathrm{BC}(\mathbb{R}^d)$ such that the first term of the right hand side is smaller than $\frac{\varepsilon}{2}$ and $N \geq 1$ such that for $n \geq N$ the second term is smaller than $\frac{\varepsilon}{2}$ by (3.24), we obtain the weak convergence of $r_\mu^{\sigma^n}$ to r_μ^σ.

Finally we show that

$$\lim_{n\to\infty} \int_{\mathbb{R}^d} |r_\mu^{\sigma^n}(x)|_2^2 \, d\mu(x) = \int_{\mathbb{R}^d} |r_\mu^\sigma(x)|_2^2 \, d\mu(x).$$

We have $|W_2(\mu, \sigma^n) - W_2(\mu, \sigma)| \leq W_2(\sigma^n, \sigma) \to 0$ as $n \to \infty$. Therefore $W_2^2(\mu, \sigma^n) \to W_2^2(\mu, \sigma)$, i.e.

$$\int_{\mathbb{R}^d} |r_\mu^{\sigma^n}(x) - x|_2^2 \, d\mu(x) \to \int_{\mathbb{R}^d} |r_\mu^\sigma(x) - x|_2^2 \, d\mu(x).$$

Since $2\int_{\mathbb{R}^d} \left\langle r_\mu^{\sigma^n}(x), x \right\rangle \, d\mu(x) \to 2\int_{\mathbb{R}^d} \left\langle r_\mu^\sigma(x), x \right\rangle \, d\mu(x)$, we have

$$\int_{\mathbb{R}^d} |r_\mu^{\sigma^n}(x)|_2^2 \, d\mu(x) \to \int_{\mathbb{R}^d} |r_\mu^\sigma(x)|_2^2 \, d\mu(x).$$

This completes the proof of Lemma A4.1. □

Proof of Proposition A4.1.

Step 1. $\sigma \in \mathcal{P}_2^a(\mathbb{R}^d)$ with compact support.

Let $\varphi : \mathbb{R}^d \to \mathbb{R}$ be the convex function of Theorem 2.1(iii). Since $\varphi \in \mathrm{Lip}(\mathbb{R}^d)$, $\varphi \in L^1(\mu; \mathbb{R}^d)$ ($|\varphi(x)| \leq |\varphi(x_0)| + C|x - x_0|_2$), and $\nabla\varphi \in L^2(\mu; \mathbb{R}^d)$ ($\nabla\varphi$ is Borel measurable essentially bounded), we can invoke Lemma 2.1.

Step 2. $\sigma \in \mathcal{P}_2^a(\mathbb{R}^d)$ with bounded density.

We can approximate σ by a sequence $\sigma^n \in \mathcal{P}_2^a(\mathbb{R}^d)$ with compact support in the $W_2(\cdot, \cdot)$ metric as it is done in Section 3.3. Then we can apply Lemma A4.1, since $\mathrm{Tan}_\mu(\mathcal{P}_2(\mathbb{R}^d))$ is a closed subspace of $L^2(\mu; \mathbb{R}^d)$.

Step 3. $\sigma \in \mathcal{P}_2^a(\mathbb{R}^d)$.

We can approximate σ by a sequence $\sigma_n \in \mathcal{P}_2(\mathbb{R}^d)$ with bounded density as it is done in Section 3.3. Then we apply Lemma A4.1.

Step 4. $\sigma \in \mathcal{P}_2(\mathbb{R}^d)$.

We approximate σ by $\sigma_n := S_\beta(\frac{1}{n})\sigma \in \mathcal{P}_2^a(\mathbb{R}^d)$ in the $W_2(\cdot, \cdot)$ metric and apply Lemma A4.1. This completes the proof of Proposition A4.1. □

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