Fine Properties of Stochastic Evolution Equations and Their Applications

Wei Liu (wei.liu@uni-bielefeld.de)

International Graduate College "Stochastics and Real World Models"
Bielefeld University

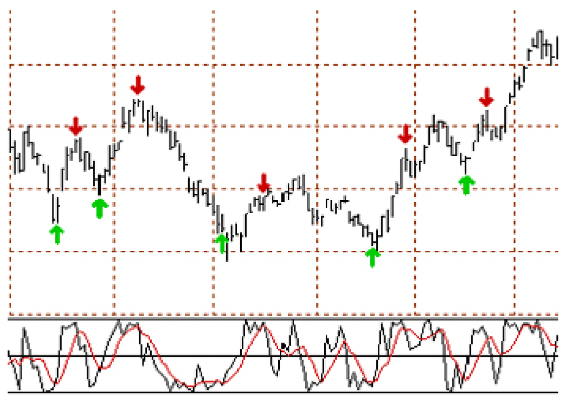
1. Introduction to SPDE

Stochastic (partial) differential equations (SPDE) have very important applications in many fields such as economics, finance, biology, physics and social sciences. For example, all kinds of dynamics with stochastic influence in nature or man-made complex systems can be modeled by stochastic evolution equations, i.e. SPDE of evolutionary type. We can formulate some concrete examples in applications:

• Application in Finance
Many researchers aim at a better understanding of stochastic evolution of financial markets through the study of appropriate mathematical models using S(P)DE. For instance, the dynamics of short interest rate X can be modeled by the following equation

$$dX_t = k(r - X_t)dt + \sigma\sqrt{X_t}dW_t,$$

where W_t is a Brownian Motion which models the random factor and k,r,σ are some parameters.



• Application in Physics and Chemistry
In physics, one can use SPDE to model the evolution process of some experiment.
(i) Stochastic porous media equations

$$dX_t = \triangle(X_t^m)dt + B(X_t)dW_t,$$

where $1 < m$ is a constant and W_t is a cylinder Wiener process on a separable Hilbert space. The above equation is a stochastic version of the classical porous media equation (i.e. $B \equiv 0$) which describes the flow of ideal gas in some porous media, here the solution X_t denotes the density of the gas and $B(\cdot)dW_t$ describes some random noise in the model.

(ii) Stochastic reaction-diffusion equations

$$dX_t = (\Delta X_t - c|X_t|^{p-2}X_t)dt + B(X_t)dW_t,$$

where $1 \le p$ is a constant and W_t is a cylinder Wiener process on a separable Hilbert space. This equation can be used to model transport phenomena in chemistry, population dynamics, transmission lines and flame propagations etc.

• Application in Biology and Social Sciences
SPDE has been also used in population genetics to model the changes in the structure of populations in time and space. For instance, we can propose the following equation

$$dX(t,\xi) = a\Delta X(t,\xi)dt + b\sqrt{X_+(t,\xi)}dW_t, \xi \in \mathbf{R}^d$$

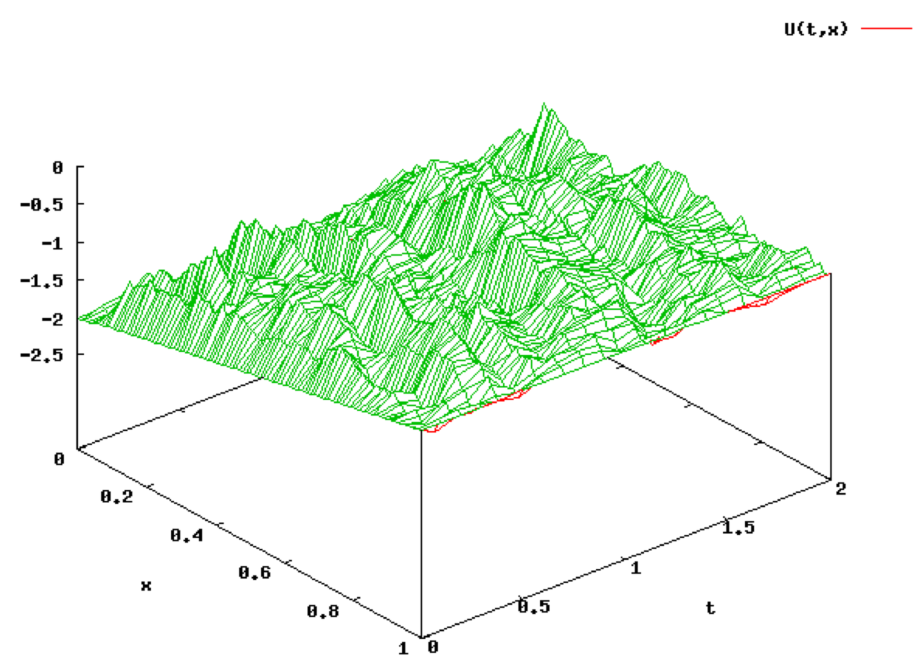
for the mass distribution $X(t,\cdot)$ of the population at time $t \ge 0$. By studying this equation we can make some prediction about the population structure in the future and make some improvement.

2. Variational Approach for SPDE

In general, we can formulate stochastic evolution equations as follows

$$dX_t = A(t,X_t)dt + B(t,X_t)dW_t. \quad (1)$$

There basically exist three different approaches to analyze SPDE in the literature: martingale approach, semigroup approach and variational approach. In the practical applications, one can use numerical analysis to simulate the solution of S(P)DE. The following graph is the simulation to the solution of a specific stochastic heat equation.



In this work we will use the variational approach, i.e. the coefficients A and B satisfy some monotone and coercive conditions such that (1) covers a large class of quasilinear and nonlinear SPDE. The existence and uniqueness of solution to (1) was established in [1]. The main aim of this work is to obtain some useful properties of the solution and some related quantities associated with SPDE.

3. Main Results and Their Applications

• Large deviation principle (LDP)
Consider (1) driven by small noise, i.e.

$$dX_t^\varepsilon = A(t,X_t^\varepsilon)dt + \varepsilon B(t,X_t^\varepsilon)dW_t, \ X_0^\varepsilon = x. \quad (2)$$

Let $\varepsilon \to 0$, then "$X^\varepsilon \rightharpoonup u$", where u satisfies

$$\frac{du_t}{dt} = A(t,u_t), \ u_0 = x.$$

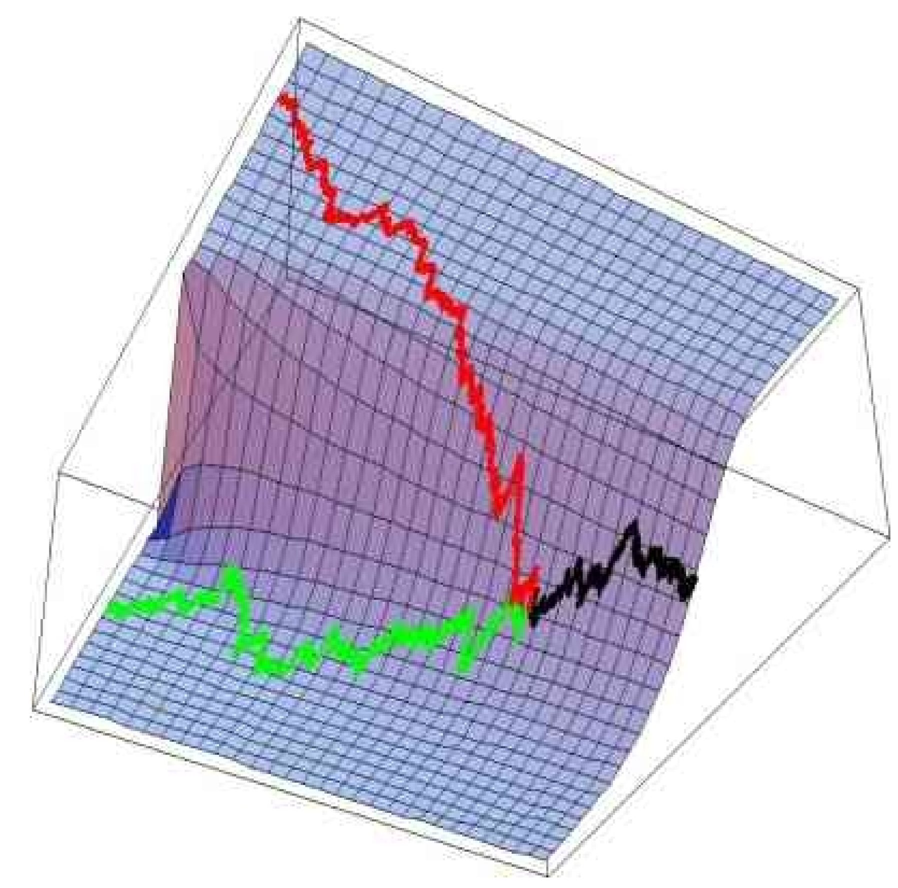
In [2] the Freidlin-Wentzell type LDP is established for (2) by using a weak convergence approach. Roughly speaking, the LDP means the rate of convergence above is exponentially fast. The main results are applied to derive the LDP for stochastic reaction-diffusion equations, the stochastic p-Laplace equation, stochastic porous media and fast diffusion equations in [2], which also improved some well-known results in the literature.

• Harnack Inequality and its applications:
By using a coupling argument and Girsanov transformation (cf.[3,4]), the Harnack inequality

$$(P_tF(y))^p \le P_tF^p(x)C(p,t,x,y), \ \ \forall F \ge 0$$

and the strong Feller property are established for the transition semigroup $\{P_t\}$ associated with (1) driven by additive type noise. Moreover, based on this Harnack inequality, we establish many resulting properties as follows:
(i) the existence, uniqueness and concentration property of invariant measures;
(ii) the ergodicity, contractivity (hyperbounded or ultrabounded) and compactness of the associated transition semigroup. In particular, we give a very easy proof for the (topological) irreducibility by using the Harnack inequality.
(iii) the convergence rate of the transition semigroup to its invariant measure, which also implies the decay estimate of the solutions to the corresponding deterministic evolution equations (e.g. p-Laplace equation, porous media equations).
(iv) some exponentially ergodicity of the transition semigroup and the existence of a spectral gap.

These results are applied to stochastic porous media equations, stochastic reaction-diffusion equations and the stochastic p-Laplace equation (cf.[3,4,5]) as examples. The main idea of coupling argument is to make two processes which start from different points to move together before certain time by adding some external force, e.g. see the following graph.



• Better smoothness of trajectory
In stochastic control and filter theory, one needs that the process has more smooth trajectory for some applications. In this part we investigate the invariance of subspaces under the solution flow of SPDE. Under some assumptions, we prove that the solution to (1) takes values in some suitable subspace of the state space if the initial state does so. This gives some stronger regularity estimates for the solution of SPDE, which can be used for some further study of the corresponding random dynamical systems (e.g. the existence of random attractor).

References

[1] N.V. Krylov and B.L. Rozovskii: *Stochastic evolution equations*, J.Soviet Math., 1981.

[2] W. Liu: *Large Deviations for Stochastic Evolution Equations with Small Multiplicative Noise*, Appl. Math. Optim., 2009.

[3] W. Liu and F.-Y. Wang: *Harnack Inequality and Applications for Stochastic Fast Diffusion Equations*, J. Math. Anal. Appl., 2008.

[4] F.-Y. Wang, *Harnack Inequality and Applications for Stochastic Generalized Porous Media equations,* Ann. Probab., 2007.

[5] C. Prévôt and M. Röckner, *A Concise Course on Stochastic Partial Differential Equations,* LNM 1905, 2007.